University of
Waterloo Research Institute

Probabilistic Analysis of Railway Tracks

Prepared for
Transportation Development Centre
Transport Canada

By
H. A. El-Ghazaly
F. Arbabi
A.N. Sherbourne

November 1985

CA1
DTC 801
85
P61
c.2

86-05786/

TP 7294
WRI No. 211-20

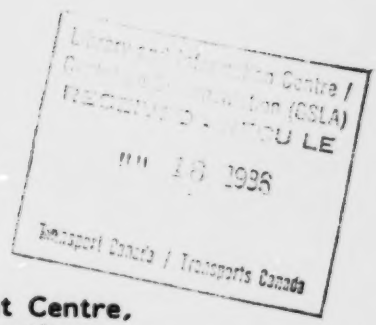
PROBABILISTIC ANALYSIS OF RAILWAY TRACKS

by

H. A. El-Ghazaly[1]

F. Arbabi[2]

A. N. Sherbourne[3]



A report prepared for the Transportation Development Centre, Transport Canada, through the Waterloo Research Institute, University of Waterloo, Ontario, N2L 3G1

November 1985

(1) Research Assistant Professor of Civil Engineering, University of Waterloo.

(2) Visiting Associate Professor of Civil Engineering (1983-84), University of Waterloo; normally Associate Professor of Civil Engineering, Michigan Technological University, Houghton, MI 49931.

(3) Professor of Civil Engineering, University of Waterloo.

The contents of this report reflect the views of the authors and not necessarily the official views of the Transportation Development Centre of Transport Canada.

PROJECT TEAM

The Project Team was composed of:

H. A. El-Ghazaly, Research Assistant Professor of Civil Engineering, University of Waterloo.

** F. Arbabi, Visiting Associate Professor of Civil Engineering (1983-1984), University of Waterloo; normally Associate Professor of Civil Engineering, Michigan Technological University, Houghton, Michigan 49931, U.S.A.

* A. N. Sherbourne, Professor of Civil Engineering, University of Waterloo.

* Project Director, December 1, 1983 - December 31, 1985

** Project Co-Director, December 1, 1983 - July 31, 1984

REPORT DOCUMENTATION FORM

1. Transport Canada Report No.	2. TDC Project No.	3. Recipient's Catalogue No.
TP 7249E	6012	

4. Title and Subtitle	5. Report Date
PROBABILISTIC ANALYSIS OF RAILWAY TRACKS	November 1985
	6. Performing Organization Report No. WRI No. 211-20
7. Author(s)	8. Transport Canada File No.
H.A. El-Ghazaly, F. Arbabi, A.N. Sherbourne	D1465-414
9. Performing Organization Name and Address Waterloo Research Institute University of Waterloo Waterloo, Ontario N2L 3G1	10. DSS File No. 14SD.T8200-3-3518 11. DSS or Transport Canada Contract No. OSD83-00039
12. Sponsoring Agency Name and Address Transportation Development Centre (TDC) 200 Dorchester Blvd. West Suite 601, West Tower Montreal, Quebec H2Z 1X4	13. Type of Report and Period Covered Final, Dec.31/83-Dec.31/85 14. Sponsoring Agency Code TDC 6012
15. Supplementary Notes	16. TDC Project Officer W.G. Rowan

17. Abstract

Track-related accidents cause many fatalities and several million dollars worth of damage every year. A better understanding of track behaviour, coupled with an analytical tool that could reasonably accurately predict the stress and instability of the track structure, would help improve the safety and economy of railroad operations.

In this study an analytical formulation and computer program is developed for stress and stability analysis of track under static deterministic and stochastic loads. The formulation is based on the finite element method. A buckling (bifurcation) analysis allows the determination of all instability modes, including lateral (gauge widening) and torsional buckling (rail overturning), from a zero displacement reference position.

The program allows complete flexibility in terms of rail profile and unit weight, input loads involving vertical, lateral and longitudinal components and/or temperature effects and four types of foundation restraint affecting vertical and lateral displacements and rail extension and rotation. While no attempt is made, at this time, to derive specific values or relationships between these components of load or foundation moduli, the analysis extends the conventional beam on elastic foundation model, which has heretofore found considerable application in track-related problems.

18. Key Words	19. Distribution Statement
Railroad, Rail, Track Stability, Loads, Stress, Buckling, Probabilistic Analysis, Computer Modelling	Limited number of copies available from the Transportation Development Centre

20. Security Classification (of this report)	21. Security Classification (of this page)	22. No. of Pages	23. Price
Unclassified	Unclassified	175	nil

TDC/CDT 79-005

FORMULE DE DOCUMENTATION POUR RAPPORT

1. Nº du rapport Transports Canada	2. Nº de l'étude — CDT	3. Nº de catalogue du destinataire
TP 7249E	6012	

4. Titre et sous-titre	5. Date du rapport
PROBABILISTIC ANALYSIS OF RAILWAY TRACKS	novembre 1985
	6. Nº du rapport de l'organisme exécutant: WRI No. 211-20
7. Auteur(s): H.A. El-Ghazaly, F. Arbabi, A.N. Sherbourne	8. Nº de dossier Transports Canada: D1465-414
9. Nom et adresse de l'organisme exécutant: Waterloo Research Institute, University of Waterloo, Waterloo, Ontario, N2L 3G1	10. Nº de dossier - ASC: 14SD.T8200-3-3518
	11. Nº de contrat — ASC ou Transports Canada: OSD83-00039
12. Nom et adresse de l'organisme parrain: Centre de développement des transports (CDT), 200 ouest, boul. Dorchester, Suite 601, Tour Ouest, Montréal, Québec H2Z 1X4	13. Genre de rapport et période visée: 31 déc. 1983- Rapport Final, 31 déc. 1985
	14. Code de l'organisme parrain: CDT 6012
15. Remarques additionnelles	16. Agent de projet — CDT: W.G. Rowan

17. Résumé

Chaque année, des accidents ferroviaires dus aux irrégularités du tracé des voies ferrées causent de nombreuses pertes de vie et des dégâts matériels de plusieurs millions. S'il était possible de prédire, grâce à des calculs analytiques, le comportement d'une voie ferrée sous les efforts de toutes sortes, la sécurité et la rentabilité du trafic ferroviaire s'en trouveraient grandement améliorées.

La recherche, dont ce rapport rend compte, a visé l'élaboration d'un modèle analytique informatisé permettant de définir le comportement d'une voie ferrée soumise à une charge statique par des méthodes tant déterministes que stochastiques, faisant appel pour cela à la méthode des éléments finis. L'analyse du flambement qui en résulte permet de déboucher sur une caractérisation des différents modes d'instabilité constatés, y compris le flambement sans torsion (variations de l'écartement) et avec torsion (rotation du rail), par rapport à une position de référence caractérisée par une absence d'instabilité.

Ce programme informatique permet de prendre en compte des rails de profils et de poids par unité de longueur différents, des charges variables décomposées en leurs composantes verticale, transversale et longitudinale, compte tenu ou non des températures, ainsi que quatre types de fixations dont la fonction est de s'opposer aux déplacements tri-axiaux de la voie et à la rotation des rails. Bien que la recherche n'ait pas voulu, pour le moment du moins, tirer des valeurs précises ou établir des rapports spécifiques entre les composantes des charges ou entre les modules de voie, elle emplifie le modèle classique de la poutre sur appuis élastiques qui a été largement utilisé jusqu'ici pour l'étude des phénomènes en jeu.

18. Mots-clés	19. Diffusion
Voie ferrée, rail, stabilité, charges, efforts, flambement, analyse probabiliste, modélisation par ordinateur	Le Centre de développement des transports dispose d'un nombre limité d'exemplaires.

20. Classification de sécurité (de ce rapport)	21. Classification de sécurité (de cette page)	22. Nombre de pages	23. Prix
non classifié	non-classifiée	175	aucun

CDT/TDC 79-006

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EXECUTIVE SUMMARY

Accidents due to derailment cause many fatalities and several million dollars worth of damage every year. It is, therefore, imperative to try to revise the present design rules regarding load estimation and track stability in order to help improve the safety and economy of railroad operations.

A number of investigators have studied different aspects of track behaviour. These studies can be generally divided into two categories, the first being primarily concerned with rail, fasteners and ties, while the second has been concerned mainly with the behaviour of ballast and subgrade. One of the purposes of the present investigation is to study the vertical loads acting on the track from a probabilistic point of view in order to illustrate a process to be used to show how the deformation response of the track can be calculated and interpreted statistically. The present investigation also intends to outline an integrated analytical formulation to calculate the deformations and stresses in the rail under various combinations of loads, including vertical, braking and lateral loads, and temperature effects. As loads are increased, a terminal condition may be reached and the rail may lose its stability due to lateral torsional buckling, also referred to as overturning. The developed formulation must properly provide quantitative and qualitative information indicating under what load combination buckling occurs and the shape of the buckled configuration, respectively.

To this end, a finite element computer program has been prepared to conduct a first order elastic stability analysis for an infinitely long rail acted upon by general 3-D loading at arbitrary, discrete locations along the rail. Four support components are provided along the rail, at the connection with the ties, which are longitudinal, vertical, lateral and one associated with torsional resistance of the anchorage between rail and ties. The buckling analysis allows the determination of all instability modes, including lateral (gauge widening) and torsional buckling (rail overturning), from a zero displacement reference position. The program allows complete flexibility in terms of rail profile, unit weight, input loads (including temperature effects) and foundation restraints. In order to facilitate the interpretation of the results, the computer output may be instantaneously processed in the form of easy to understand plots. Finally, the probabilistic analysis presented in this study is concluded by showing how the finite element method can be programmed in a stochastic format for the specific problem of rail analysis.

The present investigation represents a preliminary approach to the global problem of rail overturning under stochastic loads. There are, however, suggestions for further R & D such as improving the representation of the foundation reactions as well as considering the effects of rail deformations on the buckling load. Eventually, it will be possible to cast the computer program in a probabilistic format as regards predictions of stresses and deformations as well as stability limits.

SOMMAIRE ADMINISTRATIF

Les accidents ferroviaires consécutifs à des déraillements causent de nombreuses pertes de vie et des dégâts matériels totalisant plusieurs millions de dollars chaque année. Il est donc impérieux de revoir les règles de calcul concernant les charges supportées par les voies ferrées et leur stabilité en service si on veut améliorer la sécurité et la rentabilité des opérations ferroviaires.

Un certain nombre de chercheurs ont étudié différents aspects du comportement des voies ferrées. Les études en question peuvent être classées dans deux catégories générales: 1) celles concernant au premier chef la superstructure de la voie (rails, fixations et traverses), et 2) celles qui se préoccupent principalement du comportement de l'infrastructure, soit du ballast et de la plate-forme. L'un des objectifs fixés à la présente recherche était d'approcher la question des charges verticales supportées par la voie d'un point de vue probabiliste et de montrer comment cette approche pourrait permettre de calculer la déformation de la voie sous ces charges et d'en interpréter les conséquences par voie statistique. Cette recherche a également débouché sur une formule analytique intégrée utilisable pour calculer les contraintes et déformations des rails sous charges variables décomposées en leurs composantes verticale, transversale et longitudinale (effort de freinage), compte tenu ou non des températures. A un certain niveau de charge, le rail atteint un état limite au-delà duquel se manifeste une instabilité par flambement transversal avec rotation, phénomène couramment désigné rotation du rail. La formule analytique élaborée devrait pouvoir fournir des données quantitatives et qualitatives justes permettant d'une part de déterminer sous quelle combinaison de charges le flambement va se produire et d'autre par d'indiquer le mode de flambement qui est à craindre.

A cette fin, les chercheurs ont mis au point un programme de calcul informatique basé sur la méthode des éléments finis, et permettant de procéder à une analyse de stabilité élastique du premier ordre concernant un rail de longueur infinie soumis à des charges triaxiales à différents endroits choisis arbitrairement. Le programme prévoit également quatre termes figurant les fixations du rail aux traverses et représentant des efforts résistants longitudinaux, verticaux, transversaux et en rotation. L'analyse du flambement permet de chiffrer tous les phénomènes d'instabilité constatés, y compris le flambement sans torsion (variation de l'écartement des rails) et avec torsion (rotation du rail), par rapport à une position donnée référence représentant un déplacement nul. Le programme autorise la prise en compte de toute la gamme des profils de rails et poids unitaires ainsi que de toutes les charges (y compris les effets des variations des températures) envisageables, sans oublier les fixations du rail. Pour faciliter l'interprétation des résultats, le programme de traitement prévoit la transposition instantanée des

données en graphiques et diagrammes de lecture aisée. Enfin, les auteurs terminent leur présentation de l'analyse probabiliste proposée en indiquant une façon d'adapter la méthode des éléments finis à la modélisation stochastique pour les besoins spécifiques de l'analyse du comportement des rails.

La présente recherche constitue une approche préliminaire à l'étude du problème global de la rotation des rails sous charge stochastique. Elle n'en préconise pas moins certains axes de recherche et développement visant, par exemple, à mieux figurer les réactions des supports ainsi qu'à prendre en compte les effets de la déformation du rail sur la charge de flambement. On pense même pouvoir donner au programme de calcul une composante de traitement probabiliste pour ce qui concerne la prévision des efforts et déformations ainsi que des états limites de stabilité.

I. GENERAL CONSIDERATIONS

1. Introduction

A survey of track related accidents (1) from 1976 to 1980 shows a rate of about 4000 per year. During this period, almost 1000 people were injured and 15 died. The problem is often more severe when railroad cars carrying hazardous materials are involved. A large number of accidents have been due to track instability. The damaged caused by 114 derailments was reported at 6.4 million dollars in 1979. A similar survey (2) covering the years 1968 to 1974 indicates even higher losses per year.

In many other cases, the buckled tracks were corrected before the passage of trains and thus derailments were prevented. Such precautionary measures, though effective in preventing accidents, require significant maintenance cost and effort. Other measures taken by the railroad industry for preventing accidents include reducing train speeds following tamping of the ballast and, during hot summer days with lateral resistance at a low level or thermal stresses at a peak, the reduction of the dynamic effects prevents the rail stresses from reaching critical levels.

Most precautionary measures are based on the experience of railroad engineers or maintenance crews and, though in many cases effective, are not based on any quantitative evaluations. Experimental and analytical studies, on the other hand, would provide greater insight into the behaviour of track systems and would allow the development of better

methods of predicting deformations and stresses. For example, the results obtained from the measurements at the Facility for Accelerated Service Tests (FAST) explain quantitatively the problems that one must anticipate under severe temperature changes. The thermal stress in the rail, calculated from temperature variation in one summer day, was 13,000 psi. The same tests indicated that a seasonal variation of temperature in excess of 160°F is not unusual. Therefore, even when the rail laying temperature is carefully determined and controlled, high axial forces may be induced by temperature changes, especially in continuous welded rails. The added effects of braking, tracking and other forces may produce a critical condition.

Many other parameters have a significant impact on the behaviour of the track. Loading and dynamic interaction between trains and track, rail deformations, resistance of fasteners and ties to applied stresses are all of major importance. The ballast provides both vertical and lateral resistance, depending on its condition, depth and shoulder width. The latter is a major parameter in counteracting horizontal as well as lateral and torsional buckling. The variation of ballast properties under revenue traffic, though an important parameter , has been dealt with only marginally. In contrast with the increased foundation modulus, migration of the ballast under traffic may have a detrimental effect on ties, causing centre or end binding which may result in tie failure. Tamping of the ballast to ameliorate this problem would cause a substantial decrease in the lateral and vertical resistance of the ballast, until a stable condition is reached under traffic. Uneven

distribution of stresses on the roadbed caused by ballast migration as well as missing or defective ties may result in differential settlements. The effect of flat wheels and corrugated rails would produce higher dynamic load factors, thus creating high stress and stability problems. The overall modelling of the track presents a complex problem. Many investigators have studied the different aspects of this problem and have made significant contributions leading to a better understanding of track behaviour.

2. Previous Research

One of the analytical models that has been successfully used since the early days of railroad construction is a beam on an elastic foundation. Winkler (3), Shwedler (4) and Zimmerman (5) used this model to design longitudinal and cross-tie tracks. Further verification of this model was conducted (6) by Talbot and his co-workers. These studies showed that the characterization of a track as a beam on an elastic foundation yields good results for predicting the rail deformations and stresses under vertical loads, provided that the support stiffness is selected carefully. A more recent study (7) showed that locomotives produced the largest vertical loads and the smallest standard deviations. Thus a beam on an elastic foundation under a series of concentrated loads specified (8) by the AREA provides a simple but accurate model for tracks that have been under traffic for some time.

The beam on elastic foundation model has also been applied (9)

to lateral loads and is used to find the stress under normal conditions. Most problems causing derailments and other track related accidents are not caused under normal conditions. The many parameters that contribute to the critical conditions are not included in the simple model provided by a beam on an elastic foundation. There is thus an obvious need for more comprehensive models that can better describe the behaviour of the track and the effect of the different parameters. A number of investigators have studied different aspects of track behaviour. These studies can be divided into two categories, the first being primarily concerned with rail, fasteners and ties, while the second has been concerned mainly with the behaviour of ballast and subgrade.

Extensions of the beam on an elastic foundation have been used to study other problems that concern the rail and tie. Kerr has used these models to study (10) horizontal and vertical buckling of tracks under thermal and other loads. He has also extended this model to include the rotational stiffness provided by the ties. These models represent a rail on a continuous elastic foundation and therefore they cannot include the effect of missing ties or frozen roadbeds and are restricted by the limitation of a one dimensional model. More detailed analyses have employed the finite element method. The AAR model (11) uses two dimensional beam elements. The roadbed and tie stiffness is also represented by short beam elements which may have linear or non-linear behaviour. This study is not concerned with stability and other behaviour modes beyond the basic deformations.

The lateral torsional buckling of the rail, referred to as rail overturning, has also been studied by several researchers. A survey of the work in this area is presented (12) by A. Zarembski. The basic model, consisting of a rail supported by vertical, lateral and torsional springs as devised (13) by Timoshenko, was used (14) by Timoshenko and Langer in an analytical and experimental study. This model was later used by Chu and Wang in conjunction with the finite different method. However, the method did not lead to satisfactory results. Arbabi (15,16) presented a finite element formulation and solution technique for the study of rail overturning and other stability problems. Chu and Wang (7) subsequently used the same formulation and obtained improved results compared to their previous study which correlated more closely to field measurements. Torkamai, Bhatti and Zarembski (18) also used the Timoshenko model to study rail overturning under dynamic loads. However, they did not use the finite element method but a more simplified solution based on the Galerkin approximation.

In all these studies the foundation was represented by an oversimplified model consisting of a rigid base connected to the rail by elastic springs. Furthermore the limitation of these models did not allow for the inclusion of many parameters such as "off loading", i.e. the variations in wheel loads attached to the same axle, or distortion of ties in the horizontal plane caused by differences in braking and tractive forces on the two rails. Several investigators have conducted studies to better determine the effect of the roadbed on the track behaviour. Simple models were used by Talbot (19), Love (20), Salem

and Hay (21). These models, similar to the model proposed by the Japanese National Railroad (22), assume the same properties for the ballast and subgrade and neglect the effect of the ballast depth. The pyramid model (23) includes this depth effect approximately. An improved model (24) by Burmister used a layered elastic half space. As in previous studies, the latter model does not include the effect of the ballast cross-section. The different ballast parameters were included in a model (25) by Lundgren et al. Robnett and others (26,27) conducted similar studies in order to establish the properties of ballast and foundation. Two dimensional finite element models were used (28) by Thompson and Tarabji and others. These finite element models require a large number of elements to describe the foundation and thus the computer time and storage is significant. Although these predict the behaviour of the foundation better than the other methods described, they still have the limitations of two dimensional models and therefore cannot model the many aspects of a three dimensional problem. The different deformation modes of the rail, tie and fasteners are not included in these models. Three dimensional models were developed by (29) Hermann for the roadbed and by (30) Raymond and his co-workers. These models depict the three dimensional stresses but do not include other elements of the track system accurately. Because of the infinite depth of the foundation, any three dimensional finite element program for modelling the roadbed would require an excessive amount of computer time and storage.

The foundation models discussed were concerned with the static

behaviour of ballast and roadbed. One of the problems that affects the failure of ties is the redistribution of stresses exerted by the ballast. For a freshly tamped ballast, the stress distribution under the tie is nearly uniform. Under traffic, migration of the ballast and its crushing under the rail seat results. Thus, the track modulus and lateral resistance change. In the present design procedure of the tie, the reaction of the rail on an elastic foundation is integrated over the tie spacing. The ballast reaction to this force applied on the tie is found by assuming an effective length over which the pressure is assumed uniformly distributed. One common practice (31) is to use an effective length of 1/3 from each end of the tie. Raymond suggests (32) an effective length equal to the length of the tie itself while doubling the rail reaction.

Obviously a design based on such assumptions cannot be very accurate. Thus, for new tie materials an optimum design is not possible. A comprehensive solution of this problem would require a dynamic analysis including the motion of granular materials and the inelastic deformations associated with the crushing of the ballast under the rail seat as well as the settlement of subgrade. This is, of course, a very complicated problem. Some recent progress in the motion of granular materials such as the application of molecular dynamics to macroscopic particles hold promise in tackling this problem.

Another aspect of the problem is the nature of loading. No significant measurements were available until recently, when measurements were taken (7) by Prause et al. on the Florida east coast

corridor and the results compiled from FAST (33). These results allow for the establishment of the load distribution and calculation of the statistical parameters.

3. Present Investigation

The present study includes a preliminary formulation and development of a computer program for the analysis of stress and stability of railroad tracks under static deterministic and stochastic loads. Since the loads applied by train wheels to the track are random in nature, measurements at FAST and other test sites provide data for determining their distributions. These data are used in establishing the frequency density function which could allow analysis to be carried out directly for random loading. The response thus determined would not only give a better picture of track behaviour but can also be used to verify the accuracy of deterministic analysis.

The current analytical formulation extends the conventional beam on elastic foundation model to include four components of input load, including temperature, and four reactive components of foundation and rail resistance via the finite element method as follows:

1) The vertical load is a statistical measure of traffic across a control section of the track. It is dependent upon the type of traffic, i.e. passenger, freight, etc., vehicle configuration (wheel spacing) and density of loading.

2) Lateral loads arise from the nature of vehicle suspensions which give rise to mixed modes of vibration involving longitudinal,

lateral and torsional motions which change with time (coupled modes in the "hunting" problem). While this action may be absent in freight cars, lateral loads may also arise from variations and imperfections in the track gauge as well as in motion around curves and variations in superelevation between rails.

3) Longitudinal loads arise from expansion or contraction of rails due to temperature changes as well as from braking and accelerative forces.

Since all these loads are applied at the top inside corner of the head of the rail, i.e. at the point of track/wheel contact, they are eccentric to both the rail centroid and shear centre and thus profoundly affect the stresses in the rail, its buckling load and its resulting mode of instability.

The reactive springs included in the analysis are designed to model the following physical phenomena:

1) The vertical spring constant is an integrated measure of the foundation moduli of the tie (wood or concrete), the ballast subbase and the soil subgrade. It is also affected by the lateral stresses in the ballast between the ties which induce some lateral confinement to the ties in determining their individual reaction modulus.

2) The horizontal lateral springs measure not only the confinement of the tie by the ballast in the shoulder of the permanent way but also the friction between the tie and subgrade. The seating between rail and tie can also affect this value.

3) The torsional spring reflects the nature of the anchorage

between rail and tie via spikes or bolts and bedplates. With time, the holding down forces on either side of the rail may change, leading to reduced torsional resistance once rail overturning is instigated.

4) The longitudinal springs inserted between rail elements in the analysis attempt to accommodate the damping of the rail to longitudinal applied forces. This simply implies that any such stress must inevitably become zero over a finite length of rail. This longitudinal damping is also affected by the ballast between ties as well as the friction between tie and subgrade.

No attempt has been made, in this investigation, either to find specific relationships between the components of load input or to establish explicit values for the forces beyond a statistical treatment of the vertical wheel loads. If, however, the lateral and longitudinal forces can be expressed as a percentage of the gravity load it is likely that a similar statistical analysis would provide information on their mean values and standard deviations. Neither is such analysis carried out for the reactive components. To the extent that all these parameters can be expressed as deterministic mean values, the program is capable of finding the stresses in the rail and obtaining its buckling load. It does, however, allow for variations in the foundation properties along the axis of the rail but does not include the effect of deformations on the calculation of the stresses or the stability number.

The report is divided into five parts. Part II involves a statistical analysis of the wheel loads showing how distributions can

be simulated from limited data. It also shows the significant parameters to be included when deriving displacements from load data, how various statistical loads can interact with each other to yield statistical measures of deflection, and how the deflections under the rail, and hence the stresses and buckling loads, can be deduced by programming in a probabilistic format. This last point has not been explored in any depth as it is believed that more needs to be done in modelling the load and foundation variables in greater detail. Parts III and IV provide the details of the program for stress and stability and include all relevant force contributions to a bifurcation analysis. The report concludes by outlining some suggestions for extensions to the current investigation.

II. PROBABLISTIC ANALYSIS OF LOADS

1. Probabilistic Loads

Loads induced by locomotives and railroad cars are random in nature and dependent on a number of factors such as train speed, vehicle dynamics, etc. To date, analyses of railroad track have largely ignored the statistical nature of the load and used deterministic values instead. Design procedures usually use locomotive loads as given by the AREA code. A more detailed analysis requires a better representation of the loads as probabilistic values. One reason for ignoring the probabilistic nature of loads is a lack of available data; only in recent years (7,34) have load measurements become available.

1.1 Loads

Measurements were made on the Florida East Coast Railroad for vertical. lateral and rail seat loads, as well as rail seat and tie-center moments and bolt forces. The histogram for vertical rail loads is given in Figure 1. The shape of the histogram is dependent on the load interval used. For example, a comparison of two histograms, one using a 0.6 k load interval and the other 1.2 k, indicates that the shorter interval provides a better resolution, while the longer interval gives a better averaging estimate. It is interesting to note that the tail end of the curve is not very sensitive to the choice of interval. Therefore, the curve with 1.2 k load interval was chosen as the best overall representation. The ordinates of the curve shown in Figure 1 are given in Appendix A representing uncorrected (measured) values.

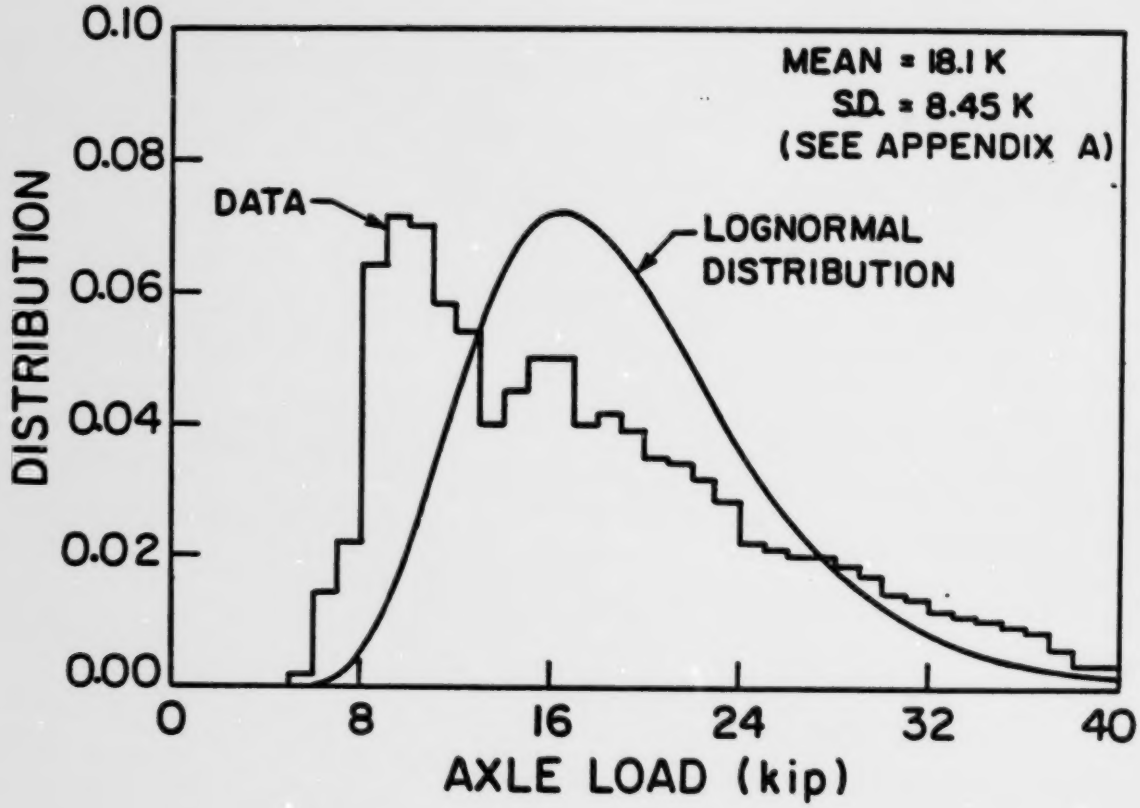


Fig. 1 Load Distribution (adjusted data)

For the measured vertical loads the following statistical values were obtained based on a sample size of 3828.

Expected value = 18.1 k

Standard deviation = 18.45 k

Coefficient of variation = 0.46

1.2 Load Distribution

A number of distributions were obtained for the load using the data of Figure 1. The best representations of the load were lognormal and Gumbel distributions, as shown in Figure 2. A normal distribution can represent the tail end of the curve accurately. It does not, however, provide a good representation of the curve at lower loads. A two curve representation may be more appropriate if normal distributions are to be used, for example, in performing a reliability analysis.

For a single curve representation of the overall data, the lognormal distribution has the advantages of accuracy and simplicity. Using the measured data given in Appendix A, the parameters a and b for this distribution will have the following values

a = 0.431894

b = 2.751055

and the distribution has the form (35)

$$f(p) = \frac{1}{\sqrt{2\pi}\, ap} \exp\left[-\frac{1}{2}\left(\frac{\ell n(p)-b}{a}\right)^2\right]$$

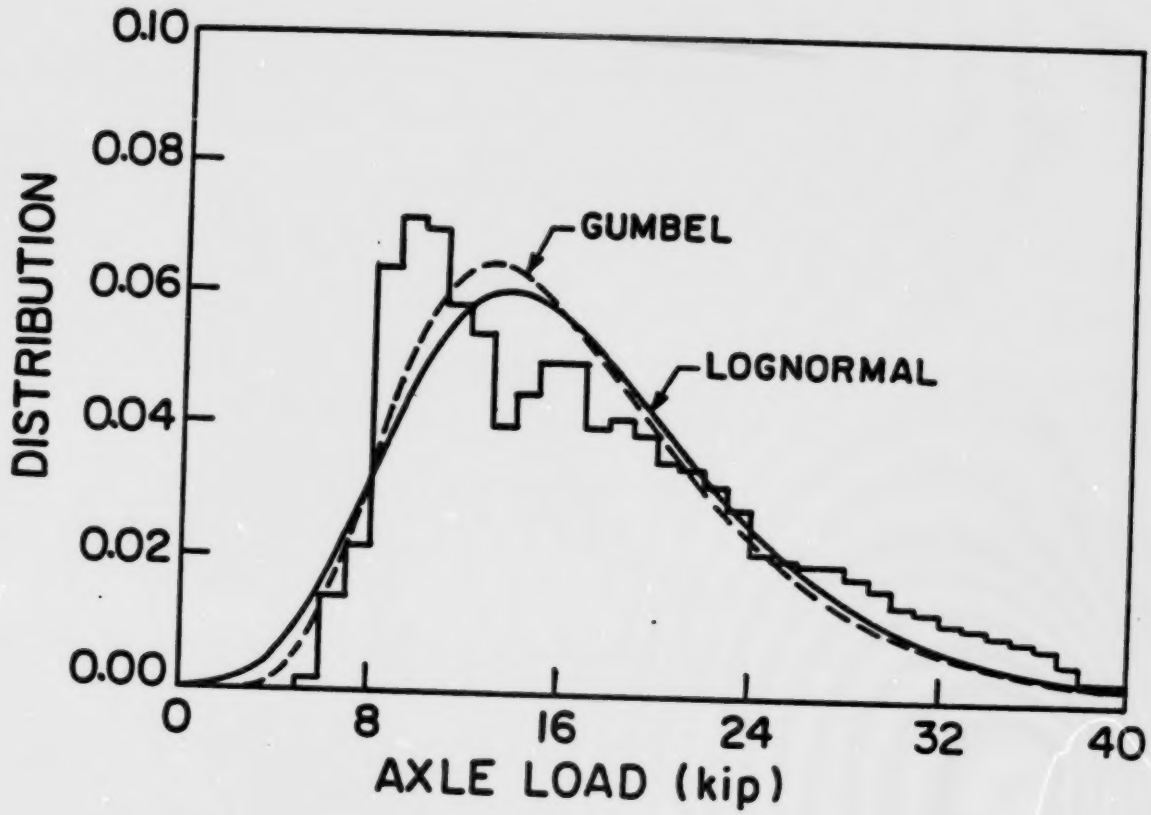


Fig. 2 Simulation

1.3 Monte-Carlo Method

In order to show the validity of the lognormal distribution for the load, a Monte-Carlo simulation of the load was performed. Random numbers with a uniform distribution were first generated using a random number generator. Using these values as cumulative distribution values, the corresponding loads were found through an assumed distribution (lognormal distribution with parameters given above). Using a sample size of 500, the mean value and standard deviation thus obtained, were 19.0 and 5.77. The details and computer program for the Monte-Carlo procedure are given in Appendix A. The latter result shows that the lognormal distribution would provide a reasonable measure of the load distribution and can be used for analysis. The advantage of using an assumed distribution over analysis done by the Monte-Carlo method with a given distribution is obvious for complex problems, since the latter method requires a multitude of analyses for different sets of random values.

In the course of the Monte-Carlo simulation it was found that the presentation in terms of selected data points, shown in Appendix A, did not give a cumulative probability of unity. Although this did not seem significant in terms of numerical values, the data points were scaled down slightly in order that the probability density function would integrate to unity. The new values resulted in the curve shown in Figure 3.

In consequence of these minor variations resulting from various degrees of simulation, unadjusted values were assumed in subsequent calculations as follows:

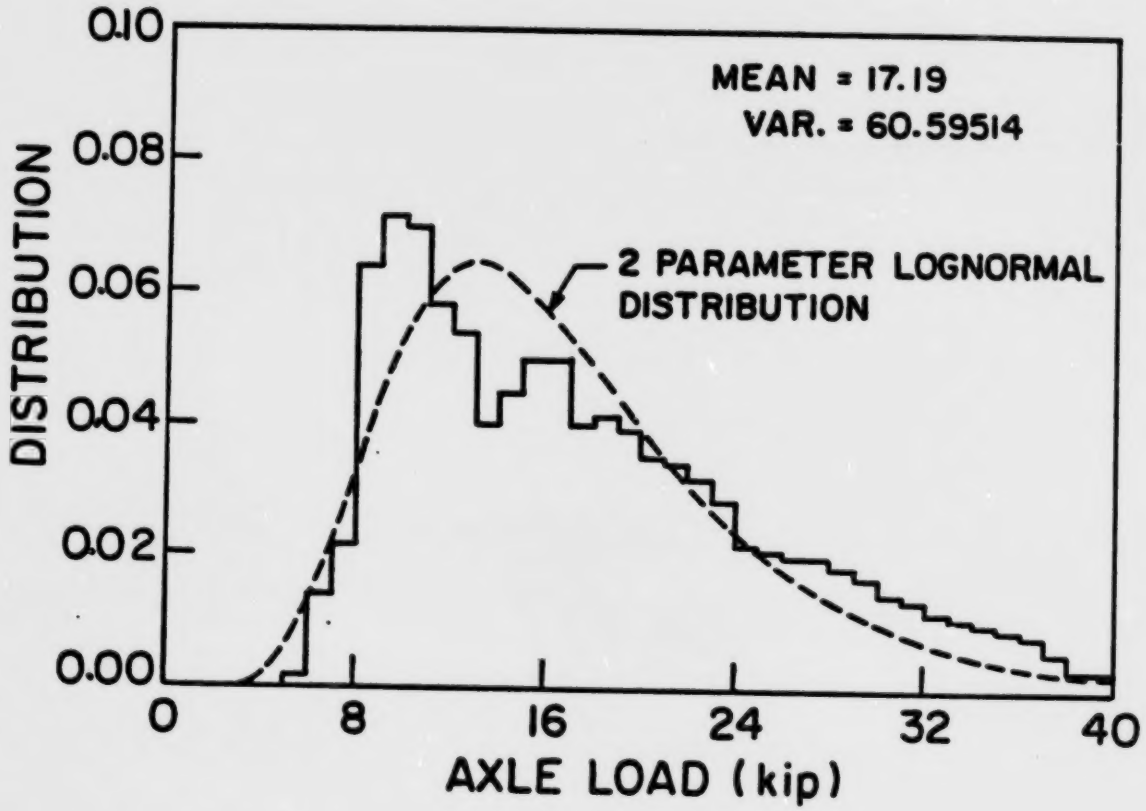


Fig. 3 Distribution for Monte-Carlo Simulation

Mean Value = 18.1 k

Std. Deviation = 8.45 k

lognormal distribution parameters, a = 0.4452, b = 2.7143

2. Probabilistic Analysis of Simple Track Models

Before embarking on the formulation of the general track problem, it is instructive to use the loads discussed in the previous section and examine some simple models which have been used in the past for the study of track. Only procedures that allow extensions to the general problem will be considered.

2.1 Beam on Elastic Foundation Model

A beam on an elastic foundation, which is one of the oldest models used for the track, is still utilized for simple designs. If the foundation modulus is known with some degree of accuracy, the results would be accurate using this model.

Considering an infinite beam with a central concentrated load, P, the deflection is

$$w = \frac{P\beta}{2U} e^{-\beta x}(\cos\beta x + \sin\beta x)$$

where $\beta = \sqrt[4]{\frac{U}{4EI}}$ and U is the foundation modulus. The maximum deflection is at the load and is equal to $w_o = \frac{P}{K} = \frac{P\beta}{2U}$. Assuming a lognormal distribution for the load, i.e.,

$$f(P) = \frac{1}{\sqrt{2\pi}\, aP} \exp\left[-\frac{1}{2}\left(\frac{\ell n(P)-b}{a}\right)^2\right],$$

the probability density function can be found for the deflection as follows:

Since $w = \alpha P = \frac{P}{K}$

with $\alpha = \frac{1}{K} = \frac{\beta}{2U} e^{-\beta x}(\cos\beta x + \sin\beta x)$

and $\beta = \sqrt[4]{\frac{U}{4EI}}$

Thus, $P = Kw$

and the probability distribution of the deflection is

$$F_W(w) = P[w \leq w] = P[P < \frac{w}{\alpha}] = F_x(\frac{w}{\alpha}) = \int_0^w \frac{\alpha}{\sqrt{2\pi}\, aw} \exp[-\frac{1}{2}(\frac{\ell n(\frac{w}{\alpha}) - b}{a})^2] \frac{dw}{\alpha}$$

or $$F_W = \frac{1}{\sqrt{2\pi}} \int_0^w \frac{1}{aw} \exp[-\frac{1}{2}(\frac{\ell n(w) - \ell n(\alpha) - b}{a})^2] \cdot dw$$

The probability density function of w is

$$f_w = \frac{1}{\sqrt{2\pi}\, aw} \exp[-\frac{1}{2}(\frac{\ell n(w) - \ell n(\alpha) - b}{a})^2]$$

For a lognormal distribution the mean and variance are

$$\bar{x} = \exp(\bar{b} + \frac{a^2}{2})$$

where $\bar{b} = \ell n(\alpha) + b$

and $V = \exp(2\bar{b}+a^2)[\exp(a^2)-1]$

Thus, for this case, all the statistical values can be obtained. Unfortunately, the determination of the distribution function is not possible except for very simple cases. Thus, in section 3 an alternative formulation is studied that can be extended and applied to complex problems.

2.2 Numerical Example

Consider a 136 lb/yd rail with a foundation modulus 365 psi and $I = 94.9\ in^4$, $E = 28{,}900$ ksi. For a load P(18.1 k, 8.45 k) and a log-normal distribution, the parameters are $a = 0.4452$ and $b = 2.7143$.

As $$\frac{1}{K} = \alpha = \frac{\beta}{2U} \quad \text{with} \quad \beta = \sqrt[4]{\frac{U}{4EI}}$$

then, $$\beta = \sqrt[4]{\frac{365\ psi}{4 \times 28900 \times 1000\ psi \times 94.9}} = 0.0135\ 1/in$$

$$K = \frac{2 \times 365}{0.0135} = 54051\ lb/in = 54.051\ k/in$$

$$\alpha = \frac{1}{K} = 0.0185\ in/k$$

The mean deflection under the load is

$$\bar{w} = \frac{18.1}{54.051} = 0.335\ in$$

$$\bar{b} = \ln(\alpha) + b = -3.9899 + 2.7143 = -1.2756$$

$$V_w = \exp(2\bar{b} + a^2)[\exp(a^2)-1]$$

$$= \exp[-2(1.2756)+(0.4452)^2][\exp(0.4452)^2-1]$$

$$= 0.0208 \text{ in}^2$$

$$\text{coefficient of variation} = \frac{(0.0208)^{1/2}}{0.335} = 0.43$$

3. Second Moment Analysis

3.1 Single Variable - Load

If the probability density function of the load is f(p), the probability of deflection is indirectly dependent on p, i.e., w = g(p). A Taylor expansion about the mean value of p = $\bar{p}$ yields:

$$w(p) = w(\bar{p}) + w'(\bar{p})\ (p-\bar{p}) + w''(\bar{p})\ \frac{(p-\bar{p})^2}{2} + \ldots \qquad (1)$$

where primes denote differentiation with respect to load, p. The question of whether the expansion should be taken about the mean or another more appropriate value is not discussed at this point. Since the expected value of $\bar{w} = E[w(p)] = \int_{-\infty}^{\infty} w(p)\ f(p)\ dp$, then using the first three terms of the Taylor expansion in w(p)

$$\bar{w} = \int_{-\infty}^{\infty} w(\bar{p})\ f(p)\ dp + \int_{-\infty}^{\infty} w'(\bar{p})(p-\bar{p})\ f(p)\ dp$$

$$+ \int_{-\infty}^{\infty} w''(\bar{p}) \frac{(p-\bar{p})^2}{2} f(p)\, dp$$

Since $w(\bar{p})$ is a fixed value and not dependent on p, the first term of each expression can be taken outside the integral sign thus

$$\bar{w} = w(\bar{p}) \int_{\infty}^{\infty} f(p)\, dp + w'(\bar{p}) \int_{-\infty}^{\infty} (p-\bar{p}) f(p)\, dp + \frac{w''(\bar{p})}{2} \int_{-\infty}^{\infty} (p-\bar{p})^2 f(p)\, dp$$

The first integral is the probability of p and is 1. The second integral is

$$\int_{-\infty}^{\infty} p\, f(p)\, dp - \bar{p} \int f(p)\, dp = \bar{p} - \bar{p} = 0$$

and the third integral is, by definition, the variance. Thus,

$$\bar{w} = w(\bar{p}) + w''(\bar{p}) \frac{\sigma_p^2}{2} \qquad (2a)$$

If the second term is neglected in the evaluation of $\bar{w}$ then the first order value of $\bar{w}$ is

$$\bar{w} = w(\bar{p}) \qquad (2b)$$

Since, in general, the deflection is a linear function of load, the last term in (2a) drops out. For the other random variables (2b) would be an approximation. To find the variance of the deflection using Equations (1) and (2b) we have

$$V_w = \sigma_w^2 = \int_{-\infty}^{\infty} (w-\bar{w})^2 f(p)\, dp = \int_{-\infty}^{\infty} [w'(\bar{p})(p-\bar{p}) + w''(\bar{p}) \frac{(p-\bar{p})^2}{2}]^2 f(p)\, dp$$

$$= \int_{-\infty}^{\infty} \{[w'(\bar{p})]^2 (p-\bar{p})^2 + [w''(\bar{p})]^2 [\frac{(p-\bar{p})^4}{4}] + w'(\bar{p})w''(\bar{p})(p-\bar{p})^3\} f(p)\, dp$$

or $V_w = [w'(\bar{p})]^2 \sigma_p^2$ + higher order terms. Thus, the first degree approximation of variance is

$$V_w = \sigma_w^2 = [w'(\bar{p})]^2 \sigma_p^2 \qquad (3)$$

3.2 Numerical Example

Applying these results to the sample problem discussed earlier in section 2.2 where

$$w = \alpha p$$

and α was found to be 0.0185, gives

$$\bar{w} = \alpha \bar{p} = 0.0185 \times 18.1 = 0.335 \text{ in}$$

and $$V_w = \sigma_w^2 = \alpha^2 V_p = (0.0185)^2 (8.45)^2 = 0.0244 \text{ in}^2$$

The coefficient of variation for w is 0.46. These results are almost exactly the same as those obtained in section 2.2. This is not surprising because, as mentioned before, the relation between the deflection and load is linear and in developing the equations (2) and (3) for mean value and variance, one retains the linear terms in the expansion of (1).

3.3 Multiple Variables - Load and Foundation Modulus

Consider again the beam on an elastic foundation and this time let the foundation, U, be a random variable as well as the load p. The displacement is given by

$$w = \frac{P\beta}{2U} e^{-\beta x}(\cos\beta x+\sin\beta x) , \quad \beta = \sqrt[4]{\frac{U}{4EI}}$$

It is reasonable to assume that the load and foundation modulus are independent random variables. The load has a lognormal distribution. In the absence of available data let us assume that the foundation modulus has a normal distribution

$$f_2(U) = \frac{1}{\sigma_U\sqrt{2\pi}} e^{-(U-\bar{U})/2\sigma_U^2}$$

where $\bar{U}$ and σ_U are the mean value and standard deviation of U. Obviously, a direct evaluation of the density function for w is rather cumbersome. We thus use an approximate procedure. Since $w = g(p,U)$, using the Taylor expansion about $\bar{p}$ and $\bar{U}$ assuming a joint probability density function for w as $f(p,U) = f_1(p)\ f_2(U)$ we have

$$g(p,U) = g(\bar{p},\bar{U}) + \frac{\partial g(\bar{p},\bar{U})}{\partial p}(p-\bar{p}) + \frac{\partial g(\bar{p},\bar{U})}{\partial U}(U-\bar{U}) + \frac{\partial^2 g}{\partial U \partial p}(p-\bar{p})(U-\bar{U})$$

$$+ \frac{\partial^2 g(\bar{p},\bar{U})}{\partial p^2}\frac{(p-\bar{p})^2}{2} + \frac{\partial^2 g(\bar{p},\bar{U})}{\partial U^2}\frac{(U-\bar{U})^2}{2} + \ldots \quad (4)$$

Then,

$$\bar{w} = E[w] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(p,U)\, f_1(p)\, f_2(U)\, dp\, dU$$

Substituting for g(p,U), the linear and cross product terms vanish and we get

$$\bar{w} = E[w] = g(\bar{p},\bar{U}) + g_{pp}\frac{\sigma_p^2}{2} + g_{UU}\frac{\sigma_U^2}{2} \quad (5a)$$

if only the second derivatives of g are retained. It should be noted that since the dependence of deflection on load is linear the second term also vanishes. The higher order term in the foundation modulus, however, does not vanish. But if this is considered insignificant, then

$$\bar{w} = g(\bar{p},\bar{U}) \quad (5b)$$

The variance is given by

$$V_w = \int_p \int_U [w-\bar{w}]^2 f(p)f(U)\, dp\, dU$$

$$= \int_p \int_U [\bar{g}_p(p-\bar{p}) + \bar{g}_U(U-\bar{U}) + \bar{g}_{pp}\frac{(p-\bar{p})^2}{2} + \bar{g}_{UU}\frac{(U-\bar{U})^2}{2} + \bar{g}_{pU}(p-\bar{p})(U-\bar{U})]f(p)f(U)dpdU$$

Considering the linear terms only, we get

$$V_w = \bar{g}_p^2 \, \sigma_p^2 + \bar{g}_U^2 \, \sigma_U^2 \qquad (6)$$

where $\bar{g}_p$ and $\bar{g}_U$ denote the derivatives of g with respect to p and U, evaluated for mean values of p and U.

3.4 Numerical Example

Consider the same example as in section 3.2, but assume that both load and foundation modulus are random values. The mean and standard deviation of the load are 18.1 k and 8.45 k. For the foundation modulus, in the absence of data, the mean value is taken as the deterministic value 365 psi with a coefficient of variation 0.1.

Since $w = \frac{P\beta}{2U} e^{-\beta x}(\cos\beta x + \sin\beta x)$ and $\beta = \sqrt[4]{\frac{U}{4EI}}$

the mean value of w will be the same as in the example of section 3.2, i.e., $\bar{w} = 0.335"$. The variance will be different due to the additional term $\bar{g}_U^2 \, \sigma_U^2$

with $$g_U = \frac{\partial w}{\partial U} = -\frac{P}{2U^2} \bar{e}^{\beta x}(\cos\beta x + \sin\beta x)$$

$$+ \frac{\partial \beta}{\partial U} \frac{P}{2U} [\bar{e}^{\beta x}(1-\beta x)(\cos\beta x + \sin\beta x) + \beta x \, \bar{e}^{\beta x}(\cos\beta x - \sin\beta x)]$$

where $$\frac{\partial \beta}{\partial U} = \frac{U^{-3/4}}{4\sqrt[4]{4EI}}$$

At the location of the load x = 0 and with β = 0.0135 1/in, U = 0.365 ksi

$$\bar{g}_U = -\frac{P\beta}{2U^2} + \frac{U^{-0.75}}{4\sqrt[4]{4EI}} \cdot \frac{P}{2U}$$

$$= -\frac{(18.1)(0.0135)}{2(0.365)^2} + \frac{(0.365)^{-0.75}}{4(4x28900x94.9)^{0.25}} \cdot \frac{(18.1)}{2(0.365)}$$

$$= -0.6877 \text{ in}^3/\text{k}$$

$$\sigma_U = (0.365)(0.1) = 0.0365 \text{ k/in}^2$$

$$V_w = \bar{g}_p \, \sigma_p^2 + \bar{g}_U \, \sigma_U^2$$

$$= (0.0185)^2 \, (8.45)^2 + (-0.6877)^2 \, (0.0365)^2$$

$$= 0.0244 + 0.0006$$

$$= 0.025 \text{ in}^2$$

Thus, the variation in foundation modulus has a negligible effect on the overall response. Even for a coefficient of variation of the foundation modulus equal to 0.46 (the load value), the variance, V_w, becomes 0.0377, resulting in a response coefficient of variation equal 0.58. Therefore, the response is not very sensitive to variation in the foundation modulus.

3.5 Discretized Response

In the solution of a discretized problem, the response is required at several points. For the sake of clarity we examine, in the following discussion, the case of a single load before discussing the general case with multiple loads.

Consider a simply supported beam with a concentrated load acting at 1/3 of the span

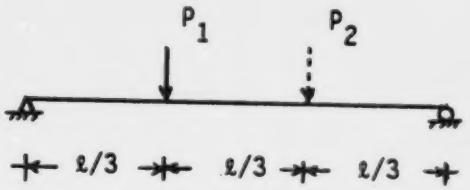


Fig. 3

The deflection at the load is

$$w_1 = \frac{4\ell^3}{243EI} P_1 = \alpha_1 P_1$$

and at $2\ell/3$

$$w_2 = \frac{7\ell^3}{486EI} P_1 = \alpha_2 P_1$$

The mean and variance of the above deflections are

$$\bar{w}_i = \alpha_i P_1 \quad \text{and} \quad V_i = \alpha_i^2 \sigma_{P_1}^2$$

where i = 1,2 indicates the point at which deflection is calculated. The covariance is

$$\mathrm{Cov}(w_1,w_2) = E[w_1-\bar{w}_1)(w_2-\bar{w}_2)] = E[\alpha_1(P_1-\bar{P}_1)\ \alpha_2(P_1-\bar{P}_1)]$$

$$= \alpha_1\ \alpha_2\ E[(P_1-\bar{P}_1)^2] = \alpha_1\ \alpha_2\ \sigma_{p_1}^{\ 2}$$

and the correlation coefficient

$$\rho_{12} = \frac{\alpha_1\ \alpha_2\ \sigma_{p_1}^2}{\alpha_1\ \alpha_2\ \sigma_{p_1}^2} = 1$$

Thus the covariance and correlation coefficient matrices would become (35)

$$\underline{C}_w = \begin{bmatrix} \alpha_1^{\ 2} & \alpha_1\ \alpha_2 \\ \alpha_1\ \alpha_2 & \alpha_2^{\ 2} \end{bmatrix} \sigma_{p_1}^{\ 2} \quad \text{and} \quad \underline{\rho} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The form of the correlation coefficient matrix indicates complete dependence of w_2 on w_1. This is obvious because

$$w_2 = \frac{\alpha_2}{\alpha_1}\, w_1$$

where α_i are deterministic values.

3.6 Multiple Variables

Consider now two random loads P_1 and P_2 acting at 1/3 and 2/3 of the span length respectively. The deflections at points 1 and 2 are

$$w_1 = \alpha_{11} P_1 + \alpha_{12} P_2$$

$$w_2 = \alpha_{21} P_1 + \alpha_{22} P_2$$

where α_{ij} are coefficients dependent on the beam properties. In view of Maxwell's reciprocal theorem $\alpha_{12} = \alpha_{21}$ and for this example $\alpha_{11} = \alpha_{22}$ by symmetry. Thus, in this case

$$\frac{w_2}{w_1} = \frac{\gamma P_1 + P_2}{P_1 + \gamma P_2}$$

where

$$\gamma = \frac{\alpha_{12}}{\alpha_{11}}$$

Thus, the ratio of the displacements is not independent of the random variables and therefore their dependence is not complete. The mean values of displacement are given by:

$$\bar{w}_1 = \alpha_{11} \bar{P}_1 + \alpha_{12} \bar{P}_2$$

$$\bar{w}_2 = \alpha_{21} \bar{P}_1 + \alpha_{22} \bar{P}_2$$

which results in covariance

$$C_{12} = E[(w_1-\bar{w}_1)(w_2-\bar{w}_2)]$$

$$= E[(\alpha_{11}P_1+\alpha_{12}P_2-\alpha_{11}\bar{P}_1-\alpha_{12}\bar{P}_2)(\alpha_{21}P_1+\alpha_{22}P_2-\alpha_{21}\bar{P}_2-\alpha_{22}\bar{P}_2)]$$

$$= E[\alpha_{11}\alpha_{21}(P_1-\bar{P}_1)^2+(\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})(P_1-\bar{P}_1)(P_2-\bar{P}_2)+\alpha_{12}\alpha_{22}(P_2-\bar{P}_2)^2]$$

$$= \alpha_{11}\ \alpha_{21}\ V_{P_1} + (\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})\ \mathrm{Cov}(P_1,P_2) + \alpha_{12}\ \alpha_{22}\ V_{P_2}$$

The variance $V_{w_1} = E[(w_1-\bar{w}_1)^2]$

$$=\alpha_{11}^2\ V_{P_1} + \alpha_{12}^2\ V_{P_2} + 2\ \alpha_{11}\ \alpha_{12}\ \mathrm{Cov}(P_1,P_2)$$

3.7 Covariance Matrix - General Derivation

In a linearized system, as discussed so far, the response can be written in terms of the random variables as follows

$$w_i = \sum_k \alpha_{ik}\ P_k$$

where summation is over all the random variables, i.e., k = 1,2, ... N. Then, the covariance of the response becomes

$$(C_w)_{ij} = E[(w_i-\bar{w}_i)(w_j-\bar{w}_j)]$$

Substitution of the response in the expression for covariance gives

$$(C_w)_{ij} = E[(\sum_k \alpha_{ik} P_k - \sum_k \alpha_{ik} \bar{P}_k)(\sum_\ell \alpha_{j\ell} P_\ell - \sum_\ell \alpha_{j\ell} \bar{P}_\ell)]$$

$$= E[\sum_k \sum_\ell \alpha_{ik} \alpha_{j\ell} (P_k - \bar{P}_k)(P_\ell - \bar{P}_\ell)]$$

$$= \sum_k \sum_\ell \alpha_{ik} \alpha_{j\ell} C_{k\ell}$$

where $C_{k\ell}$ is the covariance matrix for the random variables. In matrix form this is written as

$$\underline{C}_w = \underline{\alpha}\, \underline{C}\, \underline{\alpha}^T$$

Thus, knowing the relation between the response and the random variables, be it load or other parameters, the expected value and the covariance matrix of the response can be evaluated.

4. Probabilistic Finite Element

The solution of complex problems is best obtained using the finite element method. The power of this method is that its application is uniform, whether one deals with a problem requiring a simple element or a complex problem using a variety of elements. In order to demonstrate the application of the procedure discussed so far we choose an element consisting of a beam on an elastic foundation. Here we only carry out a solution using a few elements. As mentioned before, the

same process is applied to the track problem, once the computer program for the solution is developed.

4.1 Beam On Elastic Foundation Finite Element

The stiffness matrix for a beam on an elastic foundation is obtained using the following equilibrium equation

$$\int_0^{\ell} (EIw^{iv}+Uw)\ \delta w\ dx = 0$$

where U is the foundation modulus and δw the variation of the deflection. Upon expressing the deflection in terms of nodal displacements,

$$r_i = \langle v_1, \theta_1, v_2, \theta_2 \rangle,$$

yields

$$w = \sum_{i=1}^{4} N_i\ r_i$$

where N_i are Legendre polynomials,

$$\underline{N} = \langle (1-3\xi^2+2\xi^3),\ \ell(\xi-2\xi^2+\xi^3),\ (3\xi^2-2\xi^3),\ \ell(-\xi^2+\xi^3) \rangle$$

with $\xi = \frac{x}{\ell}$ and ℓ the length of the element. This results in the stiffness

$$K_{ij} = \int_0^{\ell} (EIN''_i\ N''_j+U\ N_i\ N_j)\ dx$$

where primes indicate differentiation with respect to x. Simple integration would yield the stiffness matrices due to the elasticity of the beam and the foundation as

$$\underline{K}_E = \begin{bmatrix} \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & \frac{-12EI}{\ell^3} & \frac{6EI}{\ell^2} \\ & \frac{4EI}{\ell} & \frac{-6EI}{\ell^2} & \frac{2EI}{\ell} \\ & & \frac{12EI}{\ell^3} & \frac{-6EI}{\ell^2} \\ & & & \frac{4EI}{\ell} \end{bmatrix}, \quad \underline{K}_S = U\ell \begin{bmatrix} \frac{13}{35} & \frac{11\ell}{210} & \frac{9}{70} & \frac{-13\ell}{420} \\ & \frac{\ell^2}{105} & \frac{13\ell}{420} & \frac{-\ell^2}{140} \\ & & \frac{13}{35} & \frac{-11\ell}{210} \\ & & & \frac{\ell^2}{105} \end{bmatrix}$$

4.2 Numerical Example (Figure 4)

Reconsidering the problem described previously and assuming that a 48' track length is sufficient for the boundary effects to be negligible, we have, as before, E = 28,900 ksi, I = 94.9 in^4, U = 0.365 ksi and P = 18.1 k at mid-length. Because of symmetry we can model one-half of the beam as shown in Figure 4(c) using two elements, 12' long. Thus, the above matrices become

$$\underline{K}_E = \begin{bmatrix} 11.2 & 794 & -11.2 & 794 \\ & 76184 & -794 & 38092 \\ & & 11.2 & -794 \\ & & & 76184 \end{bmatrix}, \quad \underline{K}_S = \begin{bmatrix} 19.5 & 396 & 6.8 & -234 \\ & 10380 & 234 & -7785 \\ & & 19.5 & -396 \\ & & & 10380 \end{bmatrix}$$

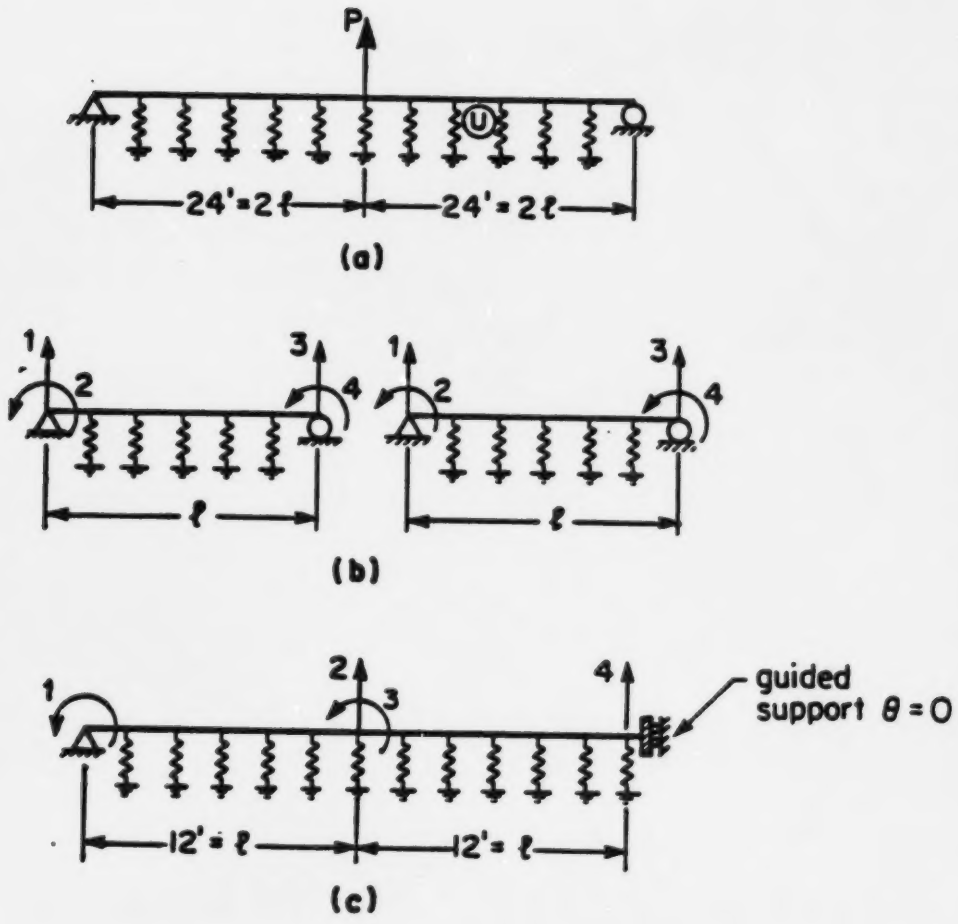


Fig.4 Finite Element Model

The element stiffness and assembled stiffness matrix for the beam are obtained as follows

$$
\begin{array}{cc} & \begin{matrix} 2 & 3 & 4 & R \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ R \end{matrix} & \begin{matrix} R \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}
\qquad
\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 2\ R\ 1 \\ 3\ 1\ 2 \\ 4\ 2\ 3 \\ R\ 3\ 4 \end{matrix} & \underline{K} = \begin{bmatrix} 30.7 & 1190 & -4.4 & 560 \\ 1190 & 86564 & -560 & 30307 \\ -4.4 & -560 & 30.7 & -1190 \\ 560 & 30307 & -1190 & 86564 \end{bmatrix} \end{array}
$$

$$
\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 86565 & -560 & 30307 & 0 \\ -560 & \begin{pmatrix} 30.7 \\ +30.7 \end{pmatrix} & \begin{pmatrix} -1190 \\ +1190 \end{pmatrix} & -4.4 \\ 30307 & \begin{pmatrix} -1190 \\ +1190 \end{pmatrix} & \begin{pmatrix} 86564 \\ +86564 \end{pmatrix} & -560 \\ 0 & -4.4 & -560 & 30.7 \end{bmatrix} \end{array}
\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ -r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -9.05 \end{pmatrix}
$$

which gives the deflection vector

$$\underline{r} = \langle 0.00024,\ -0.0205,\ -0.001,\ -0.31 \rangle$$

The displacement at the load is 0.31" compared with 0.335" calculated previously using an infinite beam. The two solutions can be made to converge as the number of elements of length ℓ is increased.

4.3 Implementation of the Probabilistic Method

In a general finite element solution the equilibrium equations are written using the assembled stiffness matrix as $\underline{K}\ \underline{r} = \underline{P}$ where, as in the above example, $\underline{r}$ is the deflection vector which is a function of random variables, i.e., $\underline{r} = \underline{r}(\underline{x})$.

In a similar manner to the procedure discussed before, we expand this deflection vector about its mean value and retain the linear terms only, thus

$$\underline{r} = \bar{\underline{r}} + \frac{\partial \bar{\underline{r}}}{\partial \underline{x}} (\underline{r} - \bar{\underline{r}})$$

From the equilibrium equation we have

$$\underline{r} = \underline{K}^{-1}\ \underline{P}$$

The expectation of this vector is

$$E(\underline{r}) = \bar{\underline{r}} = \bar{\underline{K}}^{-1}\ \bar{\underline{P}}$$

i.e., the mean values of the displacements are obtained by carrying out the solution in much the same manner as for the deterministic study, while using the mean values of all the random variables.

To find the covariance matrix, we first find the derivative of the equilibrium equation with respect to the jth random variable

$$\underline{K} \frac{\partial \underline{r}}{\partial x_j} + \frac{\partial \underline{K}}{\partial x_j} \underline{r} = \frac{\partial \underline{P}}{\partial x_j}$$

which gives

$$\frac{\partial \underline{r}}{\partial x_j} = \underline{K}^{-1} \left(\frac{\partial \underline{P}}{\partial x_j} - \frac{\partial \underline{K}}{\partial x_j}\,\underline{r}\right) \text{ with } \underline{\alpha} = \frac{\partial \underline{r}}{\partial \underline{x}} = \left[\frac{\partial \underline{r}}{\partial x_1}, \frac{\partial \underline{r}}{\partial x_2}, \cdots \frac{\partial \underline{r}}{\partial x_n}\right]$$

As already discussed, the covariance matrix would be evaluated by

$$\underline{C}_r = \underline{\alpha}\, \underline{C}_x\, \underline{\alpha}^T$$

or

$$\underline{C}_r = \left(\frac{\partial \underline{r}}{\partial \underline{x}}\right) \underline{C}_x \left(\frac{\partial \underline{r}}{\partial \underline{x}}\right)^T$$

Here $\underline{C}_x$ is the covariance matrix of the random variables. If there are m random variables and n degrees of freedom, then $\underline{C}_x$ is an m x m matrix, $\partial \underline{r}/\partial \underline{x}$ an n x m matrix resulting in an n x n matrix for $\underline{C}_r$. If the loads are the only random variables, then $\partial \underline{K}/\partial x = 0$ and the covariance matrix would be much easier to compute since, in this case, the result becomes

$$\underline{C}_r = (\underline{K}^{-1})\left(\frac{\partial \underline{P}}{\partial \underline{x}}\right)(\underline{C}_x)\left(\frac{\partial \underline{P}}{\partial \underline{x}}\right)^T (\underline{K}^{-1})^T$$

4.4 Numerical Example

Consider the example discussed in section 4.2, assuming one single random load and the foundation modulus being a random variable. Using 18.1 k and 8.45 k for mean and standard deviation of the load,

and 0.365 ksi and 0.0365 ksi for the mean and standard deviation of the foundation modulus, the mean response vector would be identical to that obtained in section 4.2. The random variables are thus U and P and are assumed to be independent. It follows that

$$\frac{\partial \underline{P}}{\partial x_1} = \frac{\partial \underline{P}}{\partial P} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{\partial \underline{P}}{\partial x_2} = \frac{\partial \underline{P}}{\partial U} = \underline{0} \text{ and } \frac{\partial \underline{K}}{\partial x_1} = \frac{\partial \underline{K}}{\partial P} = \underline{0}$$

Finally,

$$\frac{\partial \underline{K}}{\partial x_2} = \frac{\partial \underline{K}}{\partial U} = \ell \begin{bmatrix} \frac{13}{35} & \frac{11\ell}{210} & \frac{9}{70} & \frac{-13\ell}{420} \\ & \frac{\ell^2}{105} & \frac{13\ell}{420} & \frac{-\ell^2}{140} \\ & & \frac{13}{35} & \frac{-11\ell}{210} \\ & & & \frac{\ell^2}{105} \end{bmatrix} = \begin{array}{c} \\ 2R \\ 31 \\ 42 \\ R3 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & R \\ R & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 53.4 & 1085 & 18.6 & -641 \\ 1085 & 28438 & 641 & -651466 \\ 18.6 & 641 & 53.4 & -1085 \\ -641 & -651466 & -1085 & 28438 \end{bmatrix} \end{array}$$

Note that we must keep track of where these terms appear in the assembled matrix.

The inverse of the assembled matrix (last matrix in example problem section 4.2) is

$$\underline{K}^{-1} = 10^{-6}\begin{bmatrix} 15.7 & -258 & -3.16 & 68.8 \\ -258 & 20743 & 34.4 & 1820 \\ -3.16 & 34.4 & 7.86 & -258 \\ 68.8 & 1820 & -258 & 41486 \end{bmatrix}$$

$$\frac{\partial \underline{r}}{\partial x_1} = \underline{K}^{-1}\frac{\partial \underline{P}}{\partial x_1} = \begin{bmatrix} 68.8 \\ 1820 \\ -258 \\ 41486 \end{bmatrix} \times 10^{-6}, \quad \frac{\partial \underline{K}}{\partial x_2} = \begin{bmatrix} 28438 & 641 & -651466 & 0 \\ 641 & \binom{53.4}{53.4} & \binom{-1085}{+1085} & \binom{0}{+18.6} \\ -651466 & \binom{-1085}{+1085} & \binom{28438}{+28438} & \binom{0}{+641} \\ 0 & 18.6 & 641 & 53.4 \end{bmatrix}$$

$$\frac{\partial \underline{r}}{\partial x_1} = \underline{K}^{-1}\left(-\frac{\partial \underline{K}}{\partial x_2}\underline{r}\right) = -\underline{K}^{-1}\begin{pmatrix} 645 \\ -7.8 \\ -411.9 \\ -17.6 \end{pmatrix} = \begin{pmatrix} -0.012 \\ +.374 \\ 0.001 \\ 0.594 \end{pmatrix}$$

$$\underline{C}_r = \begin{bmatrix} 0.00007 & -0.012 \\ 0.0018 & 0.374 \\ -0.00026 & 0.001 \\ 0.041 & .594 \end{bmatrix}\begin{bmatrix} 72 & 0 \\ 0 & 0.00133 \end{bmatrix}\begin{bmatrix} 0.00007 & 0.0018 & -0.00026 & 0.041 \\ -.012 & .374 & 0.001 & .594 \end{bmatrix}$$

$$\underline{C}_r = \begin{bmatrix} 0.005 & -16\times10^{-6} \\ 0.130 & 0.0005 \\ -0.019 & 1.3\times10^{-6} \\ 2.95 & 0.0008 \end{bmatrix}\begin{bmatrix} 0.00007 & 0.0018 & -0.00026 & 0.041 \\ -0.012 & .374 & 0.001 & .594 \end{bmatrix}$$

The final result for the covariance matrix of displacements and rotations is

$$\underline{C}_r = \begin{bmatrix} 0.492 & 3.016 & -1.316 & 195.5 \\ 3.1 & 421 & -33.3 & 5627 \\ -1.3156 & -33.71 & 4.9 & -778.2 \\ 196.9 & 5609.2 & -766.2 & 121425 \end{bmatrix} \times 10^{-6}$$

III. STRESS ANALYSIS

1. Assumptions

The rail is modelled as a series of beam elements elastically supported at connection with ties.

a) the rail section has one axis of symmetry.

b) elastic supports are provided at the center of the rail base.

c) the rail material is linearly elastic.

d) eccentricity of the shear center w.r.t. the centroid is accounted for.

e) the external loads can be applied at any point on the cross-section and along the rail.

f) shear deformations are ignored.

g) the displacement at any point on the rail may be completely defined in terms of 7 displacement components which are $(u,v,w,\beta,\beta',\theta,\lambda)$ as in Figure 1.

h) the assumption of plane cross-sections before and after deformation remains valid, excluding warping effects.

i) the eccentricity of the support point w.r.t. the centroid and shear center has been accounted for.

2. Thin-Walled Beam Stiffness Matrix

Figure 1 shows an idealized rail section which is modelled as an I-section with unequal flanges. Reference axes and analysis data are shown. Seven degrees of freedom per node are required to describe

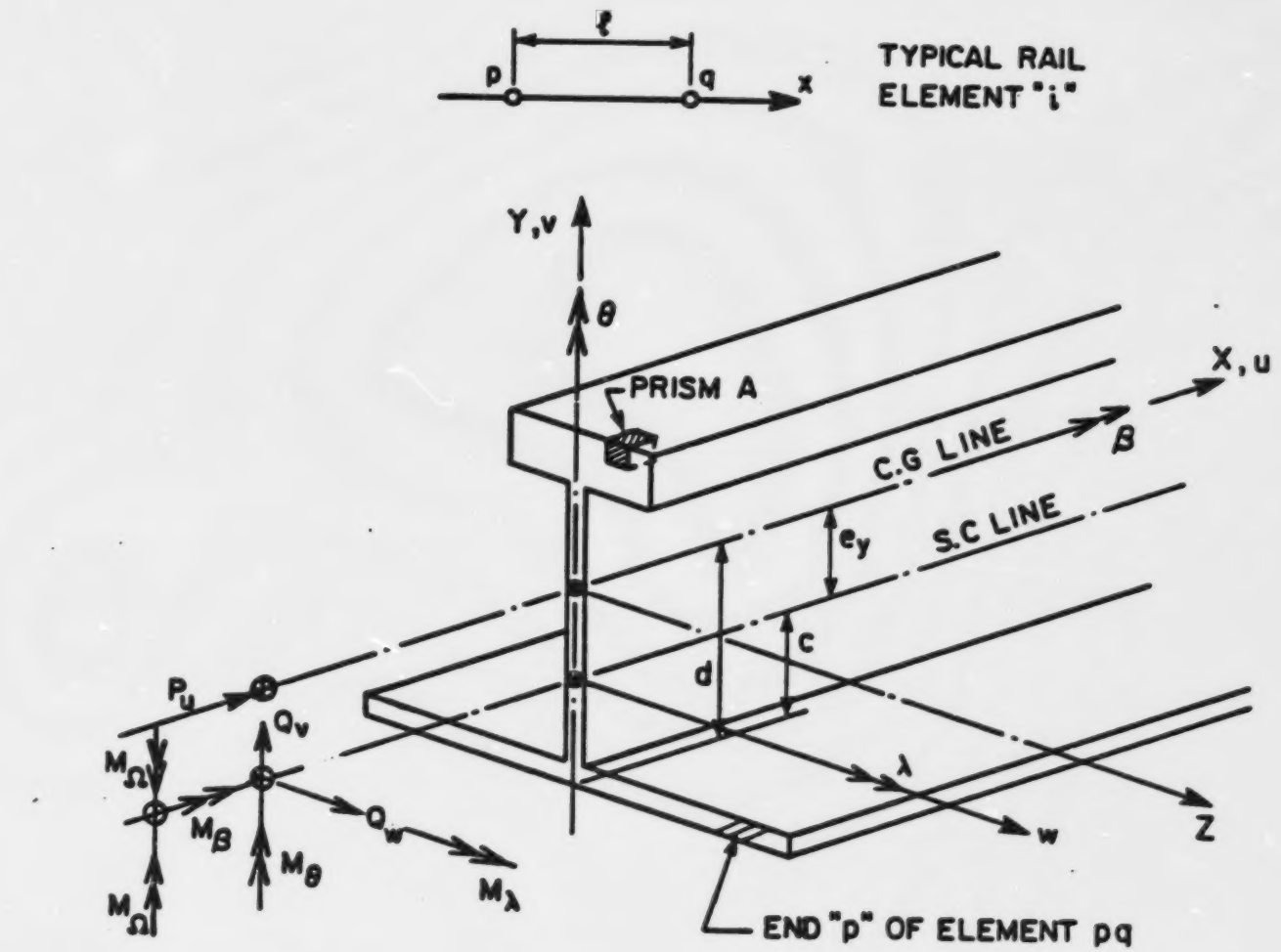


Fig. 1 Stress and Force Components

the behavior of a thin-walled beam element. The resulting element stiffness matrix is thus of dimension (14 x 14). By considering the warping rigidity an additional force component, known as the biomoment (or warping moment) M_Ω, appears. The biomoment is a self-equilibrating force when integrated over the entire cross-section and induces axial strains and stresses due to eccentric axial loads in the rails as a result of braking and acceleration forces. The resulting stiffness equation takes the form

$$[k]\{\delta\} = \{f\} \tag{1}$$

where

$$\{\delta\}^T = \{u_p, v_p, w_p, \beta_p, \beta'_p, \theta_p, \lambda_p, u_q, v_q, w_q, \beta_q, \beta'_q, \theta_q, \lambda_q\}$$

and

$$\{f\}^T = \{P^p_u, Q^p_v, Q^p_w, M^p_\beta, M^p_\Omega, M^p_\theta, M^p_\lambda, P^q_u, Q^q_v, Q^q_w, M^q_\beta, M^q_\Omega, M^q_\theta, M^q_\lambda\}$$

and [k] is the element stiffness matrix.

In order to uncouple the flexural effects ($Q_w, M_\theta, Q_v, M_\lambda$) from the torsional effects (M_β, M_Ω) it was decided to employ the displacements at both the cross-section centre of gravity (C.G.) and shear center (S.C.) as follows:

Let (u) refer to the axial displacement of the C.G., and ($v, w, \beta, \beta', \theta, \lambda$) to the displacements of the S.C. Consequently the force

vector is also split such that (P_u) refers to the axial force acting at the C.G. and (Q_v,Q_w,M_β,M_Ω,M_θ,M_λ) refer to forces acting at the S.C. For a member, as shown in Figure 1, loaded at its terminals p and q, the following homogeneous governing differential equations apply through variational analysis of the strain energy (see Part IV).

$$EAu'' = 0 \tag{2a}$$

$$EI_z v'''' = 0 \tag{2b}$$

$$EI_y w'''' = 0 \tag{2c}$$

$$EI_\Omega \beta'''' - GJ\beta'' = 0 \tag{2d}$$

where, E and G are the material Young's and Shear moduli, respectively, and

$I_z = \int_A y^2 dA$ where y is measured w.r.t. axes through the C.G.

$I_y = \int_A z^2 dA$ where z is measured w.r.t. axes through the C.G.

$J \cong \frac{1}{3} \sum_i b_i t_i^3$ where b_i and t_i are the breadth and thickness of the ith plate component

$I_\Omega = \int_A \omega^2 dA$ where ω is the sectorial area.

the prime in the equations indicates differentiation w.r.t. X.

Equations (2a-c) describe the axial and transverse equilibrium in the three directions and the associated stiffness matrices take the conventional form where a linear variation describes displacement of the C.G. in the X direction while a cubic variation describes the displacement of the shear center along the Y and Z principal axes. Equation (2d) describes torsional equilibrium and we shall elaborate on that equation.

Define

$$\kappa^2 = \frac{GJ}{E\, I_\Omega} \tag{3}$$

Then the solution to (2d) takes the form

$$\beta(x) = C_1 \cosh \kappa x + C_2 \sinh \kappa x + C_3 \kappa x + C_4 \tag{4}$$

where the integration constants can be obtained from the torsional boundary conditions

at $x = 0$ $\quad \beta = \beta_p$

$\quad \beta' = \beta'_p$

at $x = \ell$ $\quad \beta = \beta_q$

$\quad \beta' = \beta'_q$

The constants of integration are defined by the matrix relationship

$$\{C\} = [A]\{\delta\} \tag{5}$$

where

$$\{C\}^T = \{C_1\ C_2\ C_3\ C_4\}$$

$$\{\delta\}^T = \{\beta_p\ \beta'_p\ \beta_q\ \beta'_q\}$$

and [A] is defined as

$$[A] = \frac{1}{D}\begin{bmatrix} \kappa-\kappa\cosh\kappa\ell & -\kappa\ell\cosh\kappa\ell+\sinh\kappa\ell & -\kappa+\kappa\cosh\kappa\ell & -\sinh\kappa\ell+\kappa\ell \\ \kappa\sinh\kappa\ell & 1-\cosh\kappa\ell+\kappa\ell\sinh\kappa\ell & -\kappa\sinh\kappa\ell & \cosh\kappa\ell-1 \\ -\kappa\sinh\kappa\ell & 1-\cosh\kappa\ell & \kappa\sinh\kappa\ell & 1-\cosh\kappa\ell \\ \kappa-\kappa\cosh\kappa\ell+\kappa^2\ell\sinh\kappa\ell & \kappa\ell\cosh\kappa\ell-\sinh\kappa\ell & \kappa-\kappa\cosh\kappa\ell & \sinh\kappa\ell-\kappa\ell \end{bmatrix} \tag{6}$$

in which $D = 2\kappa - 2\kappa\cosh\kappa\ell + \kappa^2\ell\sinh\kappa\ell$ and ℓ refers to the element length.

Equations (4,5) completely define the torsional displacements, β,β' at any point along the rail once β, β' at the element terminals are obtained.

In order to obtain the stiffness matrix associated with torsion and warping, the torsional displacement field is defined as

$$\beta(x) = \sum_i \gamma_i(x) \qquad i = 1,2,3,4$$

Each displacement mode γ_i satisfies the governing torsional differential Equation (4) and is obtained by setting only one torsional degree of

freedom equal to unity while suppressing the remaining three degrees of freedom, e.g., to obtain $\gamma_1(x)$ set $\beta_p = 1$ and $\beta_p' = \beta_q = \beta_q' = 0$.

Employing the Principle of Virtual Work or the Galerkin method in association with the previous assumptions, it becomes possible to obtain the matrix equilibrium equation for torsion

$$[k_\beta]\{\delta_\beta\} = \{f_\beta\}$$

where

$$\{\delta_\beta\}^T = \{\beta_p,\ \beta_p',\ \beta_q,\ \beta_q'\}$$

$$\{f_\beta\}^T = \{M_\beta^p,\ M_\Omega^p,\ M_\beta^q,\ M_\Omega^q\}$$

$[k_\beta]$ is a sub-matrix that corresponds to the torsional degrees of freedom of the general thin-walled element elastic stiffness matrix and is given in Appendix B.

2.1 Definitions of the Torsion-Warping Constants

I_Ω in (2d) is known as the sectorial moment of inertia and is defined as

$$I_\Omega = \int_A \omega^2\, dA$$

where ω is the sectorial area (warping function) and is calculated for any point on a cross-section, such as D in Figure 2, as

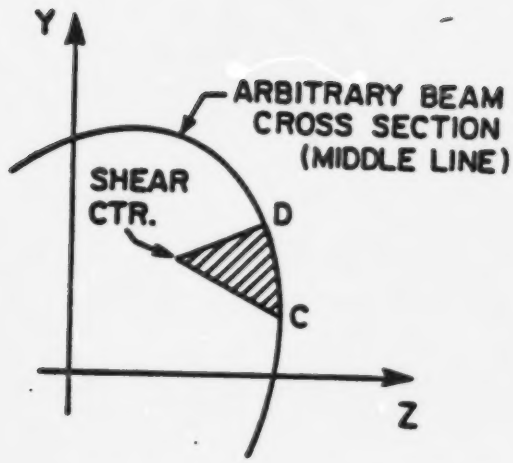


Fig. 2 Definition of Sectorial Coordinate, ω

$$\omega_D = 2 \times (\text{area OCD}) \tag{7}$$

The warping function at point D is positive if the initial radius, OC, has to rotate counter-clockwise in order to reach point D. For the sake of convenience point C is usually chosen such that

$$\int_A \omega \, dA = 0 \tag{8}$$

Any point along a cross-section axis of symmetry satisfies (8).

Figure 3 shows the distribution of ω as calculated for the rail section of Figure 4. All section properties were obtained following the idealization shown in Figure 5.

3. Spring Constants

Figure 6 shows the arrangement of the reactive elastic springs on the rail section. (k_v) represents the vertical stiffness of the subgrade including ballast and tie. For a claylike soil the mean value of k_v is about 100 kip/inch. (k_w) simulates the lateral stiffness of the surrounding soil, ballast and attachments with a mean value of about 22 kip/in for a claylike soil. The mean stiffness of the torsional spring (k_β), representing the resistance of the rail spikes or anchors, can be approximated at 100 kip.in/rad for a claylike soil. Axial springs have also been considered in the present investigation. The existence of such resistance damps out the tractive axial force acting along the rail. The typical mean value of the axial spring (k_u) may

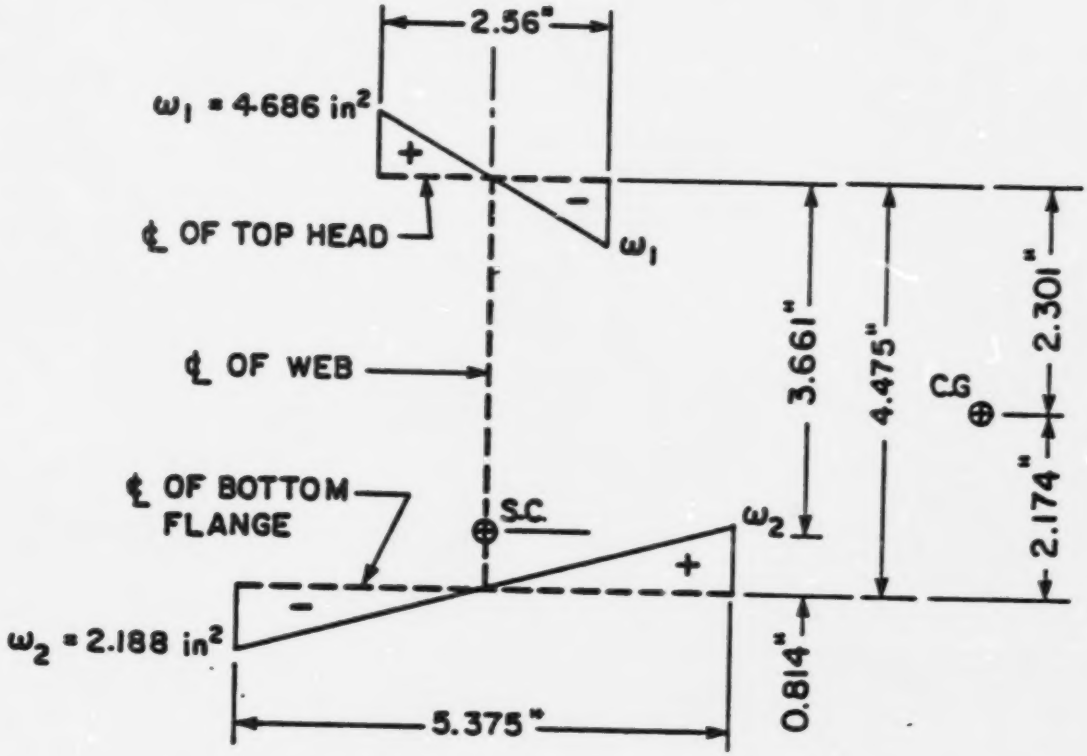


Fig. 3 Distribution of Sectorial Coordinate, ω

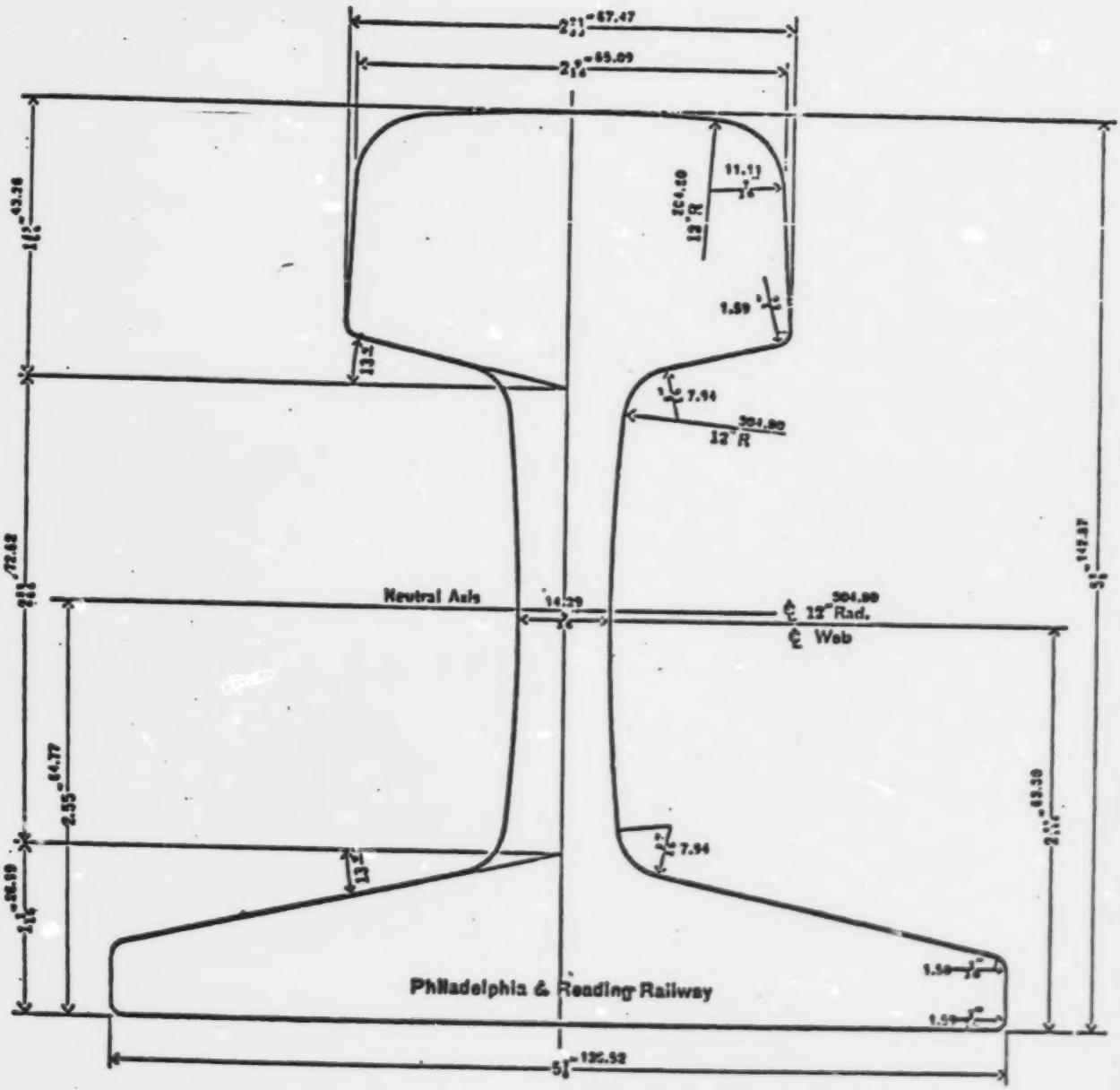
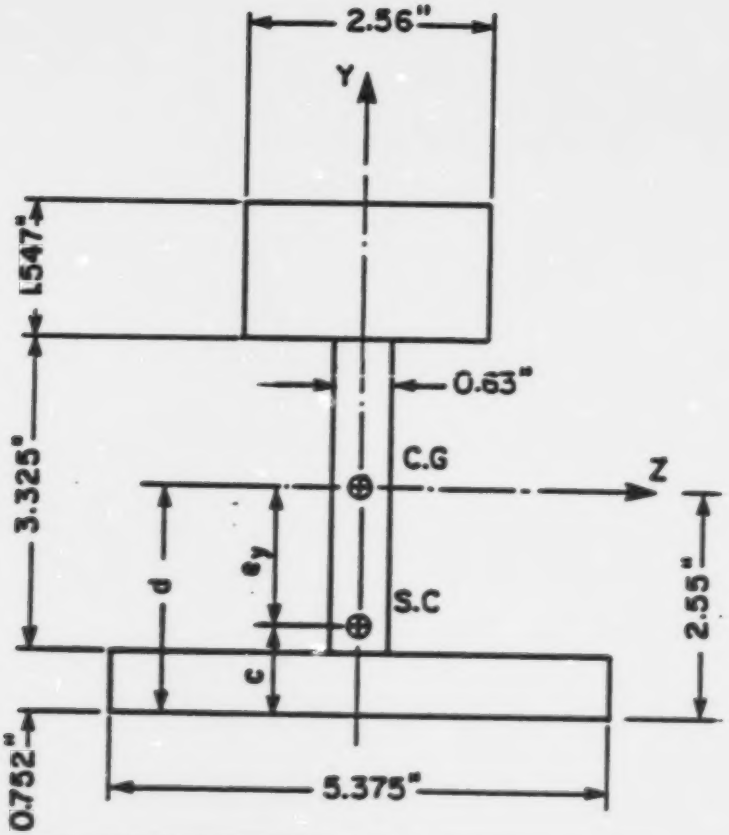


Fig. 4 Typical Rail Section



PROPERTIES:

A	= 10.1 in^2	β_1	= -110.76 in^5
I_y	= 11.96 in^4	γ	= 63.5 in^5
I_z	= 43.02 in^4	e_y	= 1.36 in
J	= 4.198 in^4	d	= 2.55 in
I_Ω	= 35.44 in^6		
I_{p_o}	= 73.78 in^4		

Fig. 5 Idealized Rail Section of Fig. 4

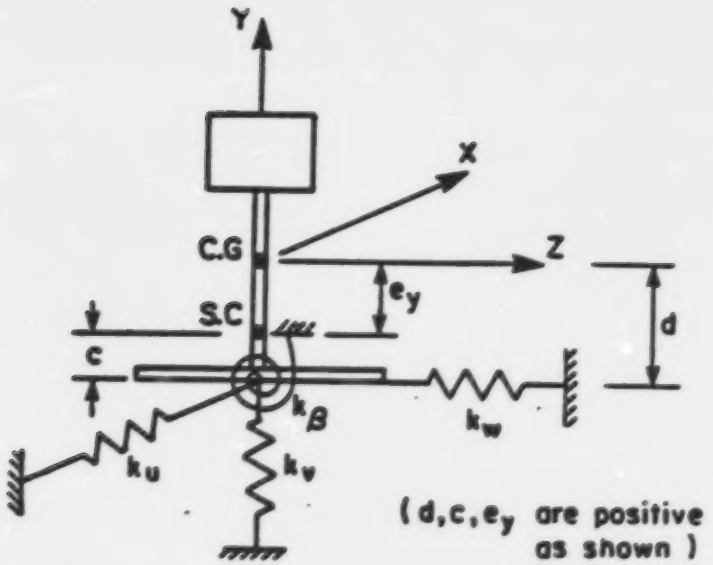


Fig. 6 Geometry of Constraints

be taken as 50 kip/in. The axial spring simulates the frictional resistance between the rail and subgrade, together with seating plates, and the resistance offered by ballast, compacted between the ties, to longitudinal movement of the ties. Although suggested mean values of k_u, k_v, k_w and k_β are used in the sample problems that follow, the program can include any assigned combination of these foundation variables or any variation of these values along the rail length.

3.1 Spring Stiffness

Springs representing soil resistance are assumed to act at the center of the rail bottom flange. To facilitate analysis, the continuous spring resistance is lumped at the locations where ties are connected to the rail. With respect to coordinates at the base of the rail the following spring stiffness matrix results

$$[\bar{k}_s] = \begin{bmatrix} k_u & & & & & & \\ & k_v & & & 0 & & \\ & & k_w & & & & \\ & & & k_\beta & & & \\ & 0 & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

where k_u, k_v, K_w and k_β represent the actual spring constants whose

values are suggested earlier. From Figure 6 the following relationships may be established relating the displacements $\{\delta\}$ at the rail centroid to displacements at the point where the springs are connected to the rail section $\{\delta_s\}$:

$$\{\delta_s\} = [T]\,\{\delta\} \tag{10}$$

In more detail, (10) reads

$$\begin{pmatrix} u_s \\ v_s \\ w_s \\ \beta_s \\ \beta'_s \\ \theta_s \\ \lambda_s \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \\ \beta \\ \beta' \\ \theta \\ \lambda \end{pmatrix} \tag{11}$$

where d and c are positive as shown in Figure 6.

The contribution of the springs to the rail stiffness, as defined at the rail centroid, is thus given by the spring stiffness matrix $[k_s]$ which is obtained as follows

$$[k_s] = [T]^T\,[\bar{k}_s]\,[T]$$

where $[k_s]$ represents the global stiffness matrix of an additional element which, in this case, is the assembly of springs at the node

and is thus incorporated into the global stiffness matrix [k] of (1). Having solved for {δ} in (1), the spring forces may be easily obtained as

$$\{f\} = [\bar{k}_s]\ \{\delta_s\}$$

where $\{\delta_s\}$ is defined in terms of {δ} which is a subset of the structural displacement vector.

4. Example

The rail section shown in Figure 5 has a total length of 660 in. where ties are spaced 22 in. apart. The local foundation constants are as follows:

$k_u = 50$ kip/in.

$k_v = 100$ kip/in.

$k_w = 22$ kip/in.

$k_\beta = 100$ kip in/rad

$k'_\beta = k_\theta = k_\lambda = 0$

The rail is subject to vertical, lateral and axial eccentric forces about the rail center as shown in Figure 7.

It is required to investigate the stresses in the rail at the

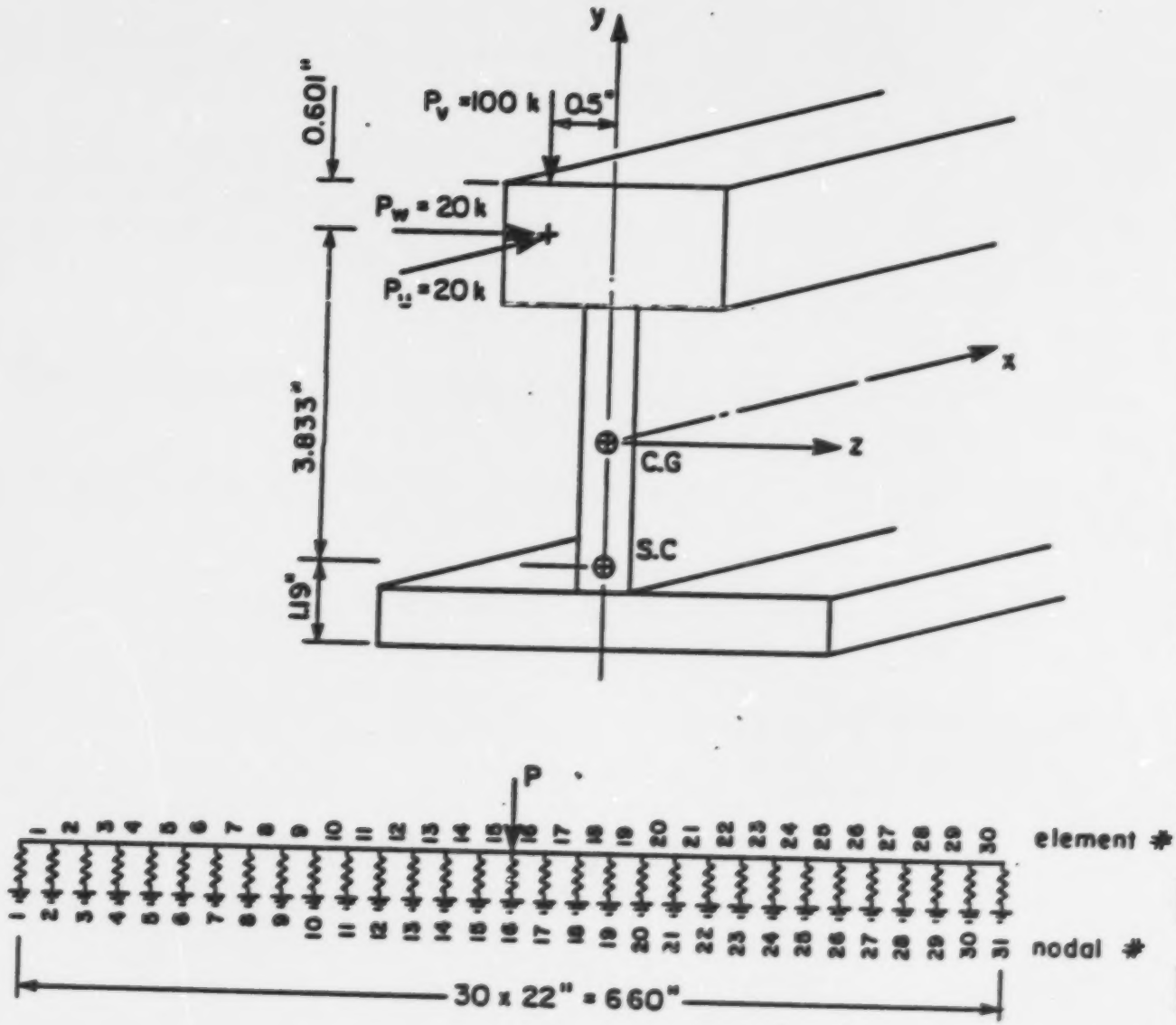


Fig. 7 External Loads on the Rail

central section due to the loads shown and also due to a temperature rise of 75°C. Assume,

$E = 28900$ ksi

$G = 11100$ ksi

4.1 Stresses Due To Load

According to the sign convention of Figure 1 and the eccentricity of the applied loads in Figure 7, the global load vector at node 16 is taken as

$P_u = +\ 20$ kips

$Q_v = -\ 100$ kips

$Q_w = +\ 20$ kips

M_β (w.r.t. shear center) $= 20(3.833) - 100(0.5) = +\ 26.66$ kip in.

M_Ω: If the axial loads are given in the form of a continuous stress distribution on the rail section, $\sigma(y,z)$, then the bimoment is defined as

$$M_\Omega = \int_A \sigma(y,z)\ \omega\ dA$$

If, however, the axial effects are concentrated point loads, the bimoment, M_Ω, is calculated as

$$M_\Omega = \sum_i P_i \omega_i \qquad (i = \text{loaded points on the x-section})$$

In the example shown, $P_i = +20$, acting at a point where $\omega_i = 2$ in^2 (figure 3) or

$$M_\Omega = 20\ (2) = 40 \text{ kip in}^2$$

Also $$M_\theta = -20\ (0.5) = -10 \text{ kip in}$$

$$M_\lambda = -20\ (2.473) = -49.46 \text{ kip in}$$

The global load vector at node 16 is thus

$$\{F_{16}\}^T = \{20., -100., 20., 26.66, 40., -10., -49.46\}$$

The finite element solution yields the following internal forces vector acting at end p of element number 16 (Figure 7).

$$\begin{bmatrix} P_u \\ Q_v \\ Q_w \\ M_\beta \\ M_\Omega \\ M_\theta \\ M_\lambda \end{bmatrix} = \begin{pmatrix} +11.306 \text{ kips} \\ -33.85 \text{ kips} \\ +6.98 \text{ kips} \\ +14.674 \text{ kip in} \\ +88.53 \text{ kip in}^2 \\ -168.24 \text{ kip in} \\ -768.04 \text{ kip in} \end{pmatrix}$$

4.1.1 Axial stresses

Axial stresses in the cross-section are induced due to (P_u, M_Ω, M_θ, M_λ). The following expression is considered in calculating the axial stresses

$$\sigma(x,y,z) = \frac{N(x)}{A} - y\,\frac{M_\lambda(x)}{I_z} - z\,\frac{M_\theta(x)}{I_y} + \frac{M_\Omega(x)}{I_\Omega}\,\omega_p(y,z) \qquad (12)$$

where $N(x) = - P_p$

$$M_\lambda(x) = - M_\lambda^p + Q_v^p(x)$$

$$M_\theta(x) = M_\theta^p + Q_w^p(x)$$

and $$M_\Omega(x) = \left[\frac{1}{\sinh \kappa\ell}\{M_\Omega^q + M_\Omega^p \cosh \kappa\ell\}\sinh \kappa x\right] - \left[M_\Omega^p \cosh \kappa x\right]$$

4.1.2 Shear stresses

Shear is caused by Q_v, Q_w and M_β. It is convenient to break up M_β as follows

$$M_\beta = M_\beta^s + M_\beta^\omega$$

where M_β^s is St. Venant torque and M_β^ω is the Warping torque. It can be shown that

$$M_\beta^s \text{ (at end p)} = - G\,J\,\beta_p'$$

$$M_\beta^\omega \text{ (at end p)} = E\,I_\Omega\,\beta_p'''$$

and finding β', β''' is relatively easy from Equation (4) once the torsional displacements become known at element terminals. At end p of element 16 the following components of M_β^s and M_β^ω were evaluated.

$$M_\beta^s = -4.198 \quad \text{kip.in}$$

$$M_\beta^\omega = +18.872 \quad \text{kip.in}$$

such that $M_\beta = M_\beta^s + M_\beta^\omega = -4.198 + 18.872 = 14.674$ kip in. M_β^s is assumed to be resisted by each plate component according to its torsional rigidity, J_i, where

$$J_i = \frac{1}{3} b_i t_i^3$$

and b_i and t_i are the plate component width and thickness, respectively. Each plate component is thus subject to a torque $M_{\beta i}^s$ such that

$$M_{\beta i}^s = M_\beta^s \frac{J_i}{\sum_i J_i} \qquad (i = 1,2,3)$$

The elastic shear stress in each plate component assumes a linear distribution across the thickness with maximum shear occurring at the outermost fibres

$$\tau_{max} = M_{\beta i}^s \frac{3}{b_i t_i^2}$$

The direction of τ agrees with the direction of $M^{S}_{\beta i}$.

Shear stresses due to $Q_v(x)$ and $Q_w(x)$ are calculated as follows

$$\sigma_{xy} = \frac{-Q_v(x) \cdot S_z(y)}{I_z \cdot t_p(y)}$$

$$\sigma_{xz} = \frac{-Q_z(x) \cdot S_y(z)}{I_y \cdot t_p(z)}$$

where $S_z(y) = \int_o^y y \, dA$

$$S_y(z) = \int_o^z z \, dA$$

and $Q_v(x) = Q_v^p$

$$Q_w(x) = Q_w^p$$

It must also be realized that M^{ω}_{β} results from two equal and opposite shear forces Q^t_w and Q^b_w acting through the top and bottom flanges, respectively. The corresponding shear stress distribution in the top and bottom flanges is given by the following expressions

$$\sigma^{top}_{xz} = \frac{-M^{\omega}_{\beta}(p)}{h} \quad \frac{S^T_y \cdot (z)^T}{I^T_y \cdot t_t}$$

$$\sigma_{xz}^{bottom} = \frac{M_\beta^\omega(p)}{h} \frac{S_y^b \cdot (z)^b}{I_y^b \cdot t_b}$$

where $S_y^T = \int_o^z z \, dA_t$ and $I_y^T = \int_{A_t} z^2 \, dA_t$

$$S_y^b = \int_o^z z \, dA_b \quad \text{and} \quad I_y^b = \int_{A_b} z^2 \, dA_b$$

A_t and A_b are the areas of the top and bottom flanges, respectively, $M_\beta^\omega(p)$ is the warping torque at end p and h is the distance between centroids of the two flanges.

Figures 8 and 9 illustrate the variation of displacements and internal forces along the rail centroid.

4.2 Stresses Due To $\Delta T = + 75°C$

The effect of temperature is to induce uniform axial strains

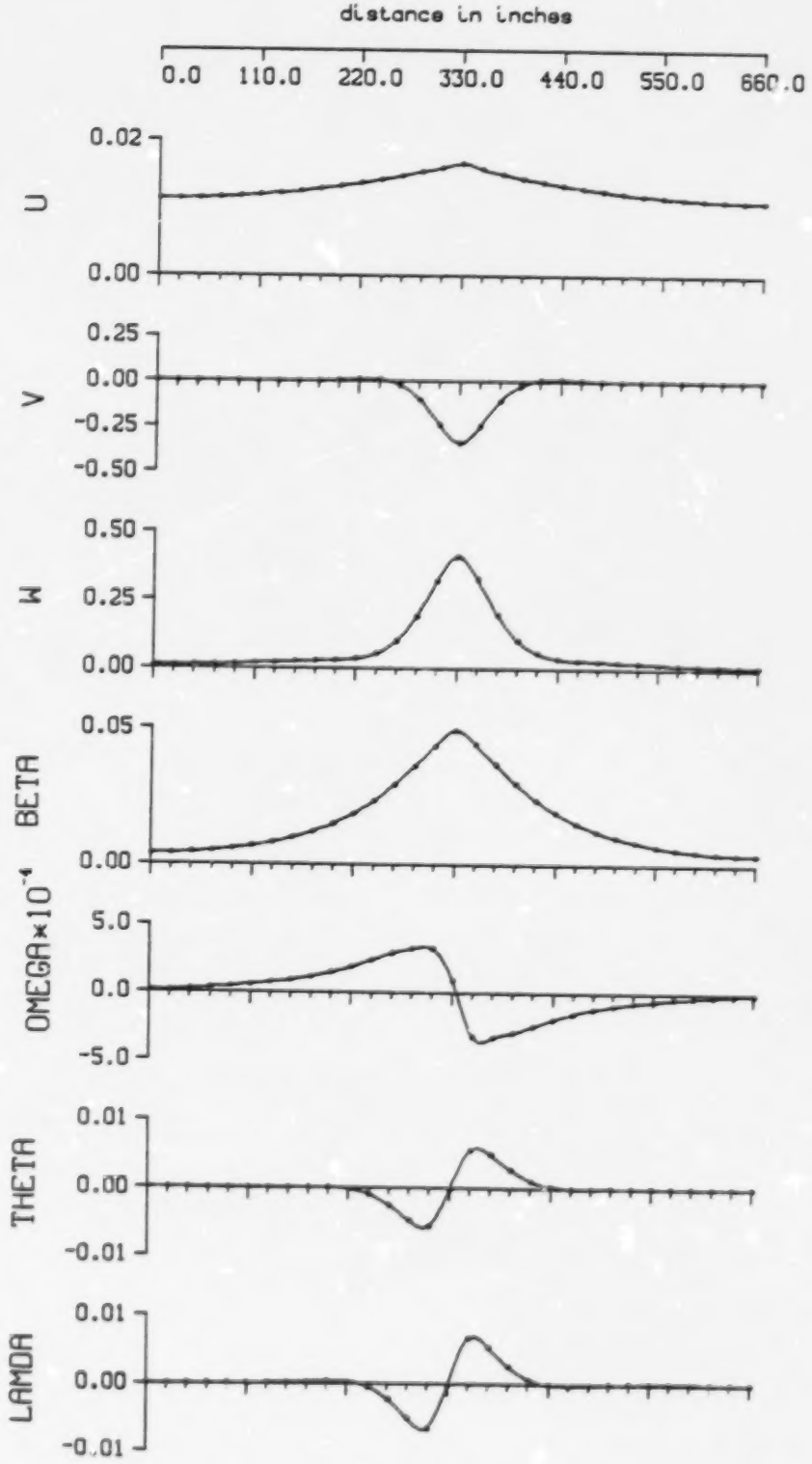
$$\varepsilon_T = \alpha \, \Delta T \qquad (\alpha \text{ assumed } 12 \times 10^{-6}/°C)$$

The axial stress is therefore uniform and of value

$$\sigma_T = E \, \varepsilon_T = (30000)(12 \times 10^{-6})(75) = 27 \text{ ksi}$$

Displacements Along Rail Centroid

(in inches ,radians and radians per inch)

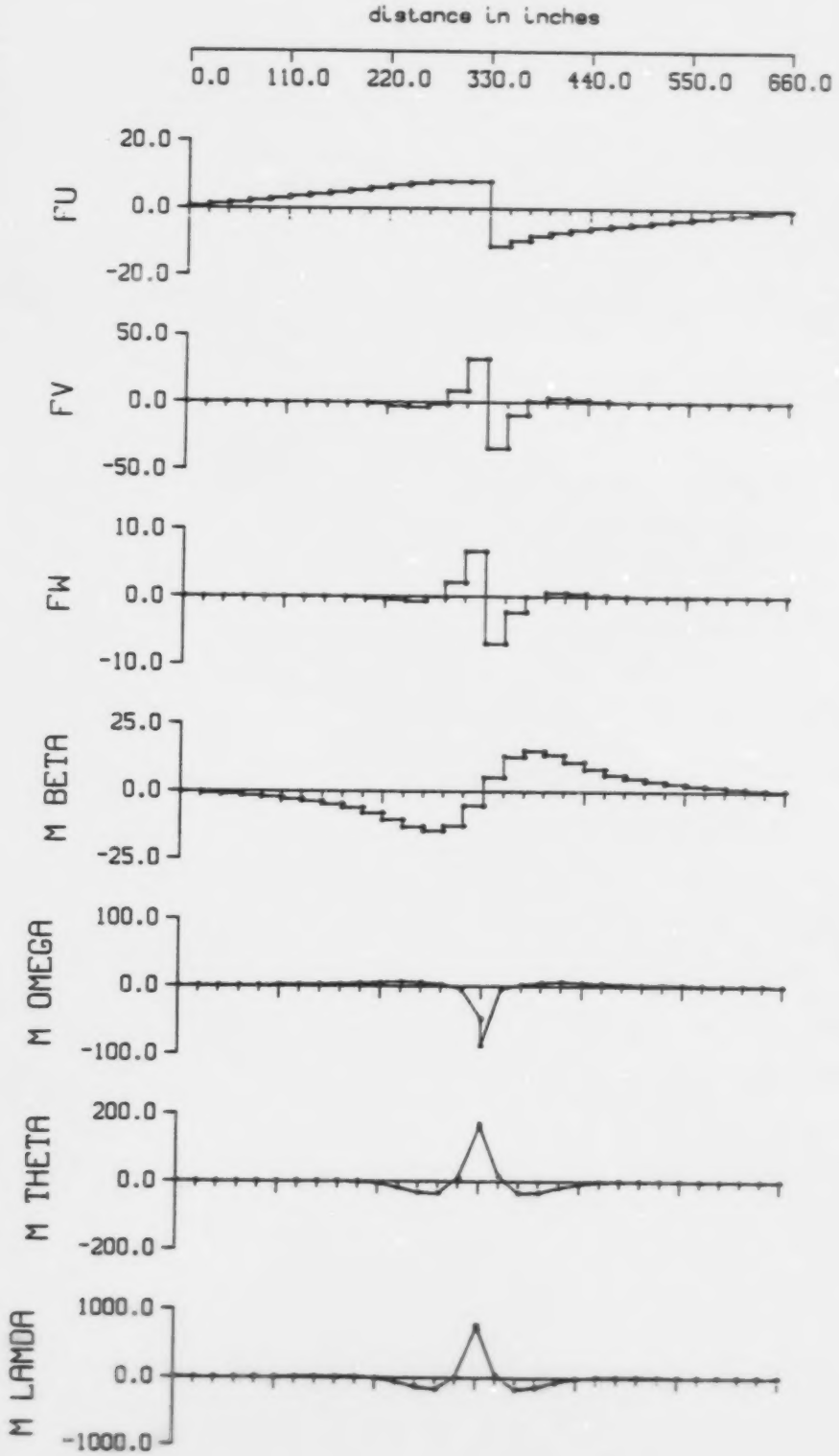


PLOT 1 15.17.55 SAT 20 DEC, 1986 JOB-RAIL , University of Waterloo DISSPLA 9.2

Fig. 8

Internal Forces Along Rail Centroid

(in Kips,Kips.inch and Kips.inch.inch)



PLOT 2 15.18.04 SAT 28 DEC, 1985 JOB-RAIL , University of Waterloo DISSPLA 9.2

Fig. 9

Because of the lack of information about the temperature gradient along the rail, the stiffness of the axial spring is assumed equal to zero for temperature analysis.

For the combined effects of loading and temperature, the final stresses due to each effect can be linearly superimposed.

IV. STABILITY ANALYSIS

According to Vlasov's theory for the analysis of thin-walled structures, the axial displacement (u) of a general point (x,y,z) on the rail cross-section is defined as

$$u(x,y,z) = U_o(x) - y\,V_{s,x} - z\,W_{s,x} + \omega_p\,(y,z)\,\beta_{,x} \qquad (1)$$

where $U_o(x)$ is the axial displacement of the centroid of the section

V_s, W_s are the displacements of the shear centre along the axes Y and Z respectively

$\omega_p(y,z)$ is the warping function of the cross-section which takes the following form for the rail section

$$\omega_p(y,z) = -(y+e_y)\cdot z \qquad (2)$$

and $\beta_{,x}$ is $(d\beta/dx)$ where β is the angle of twist of the cross-section as defined at the shear centre.

Equation (1) includes the effects of direct axial movement, double bending and warping on the total axial displacement of any point (x,y,z) on the cross-section.

As in conventional solid beam theory, the strain energy due to shear strains resulting from bending will be ignored since it is small compared to the axial strain energy. In addition, the shear strain energy due to non-uniform torsion will also be ignored for the same

reason. The shear strain energy due to uniform torsion (St. Venant torsion) must be considered since uniform torsion induces no axial strains.

1. Variational Analysis

The strain energy density μ_s^i is defined as

$$\mu_s^i = \int_0^{\varepsilon} \sigma \, d\varepsilon = \int_0^{\varepsilon} E\varepsilon_{xx} \, d\varepsilon + \int_0^{\gamma_i} G\gamma_i \, d\gamma_i \tag{3}$$

where ε_{xx} is the axial strain which, according to (1), becomes

$$\varepsilon_{xx}(x,y,z) = U_{o,x} - y V_{s,xx} - z W_{s,xx} + \omega_p(y,z) \beta_{,xx} \tag{4}$$

γ_i is the shear strain at any plate component (i) comprising the thin-walled cross-section.

$$\mu_s^i = \frac{E}{2} \varepsilon_{xx}^2 + \frac{G}{2} \gamma_i^2 \tag{5}$$

The total strain energy of the element is thus obtained as

$$U_s = \int_V \mu_s \, dV$$

$$= \frac{E}{2} \int_V \varepsilon_{xx}^2 \, dV + \frac{G}{2} \sum_i \int_{V_i} \gamma_i^2 \, dv_i \tag{6}$$

The summation over the number of plate components in (6) is correct since the strain energy is a scalar.

If the shear stresses due to St. Venant torsion assume a linear distribution across the thickness of each plate component, the second integral in (6) can be expressed as

$$\frac{GJ}{2} \int_{\ell} \beta_{,x}^2 \, dx$$

where J (St. Venant Constant) = $\frac{1}{3} \sum_i b_i \, t_i^3$ (7)

Equation (6) now reads

$$U_s = \frac{E}{2} \int_V \varepsilon_{xx}^2 \, dV + \frac{GJ}{2} \int_{\ell} \beta_{,x}^2 \, dx \tag{8}$$

Substituting (4) into (8) results in

$$U_s = \left[\frac{E}{2} \int_V \{U_{o,x}^2 + y^2 V_{s,xx}^2 + z^2 W_{s,xx}^2 + \omega_p^2(y,z) \, \beta_{,xx}^2 \right.$$

$$- 2y \, U_{o,x} V_{x,xx} - 2 z \, U_{o,x} W_{s,xx} + 2 U_o(x) \, \omega_p(y,z) \, \beta_{,xx}$$

$$+ 2yz \, V_{s,xx} W_{s,xx} - 2y \, V_{s,xx} \, \omega_p(y,z) \, \beta_{,xx}$$

$$\left. - 2z \, W_{s,xx} \, \omega_p(y,z) \, \beta_{,xx} \} \, dV \right] + \frac{GJ}{2} \int_{\ell} \beta_{,x}^2 \, dx \tag{9}$$

We note that

$$\int_A y\, f_1(x)\,dA = 0 \tag{10-a}$$

$$\int_A z\, f_2(x)\,dA = 0 \tag{10-b}$$

$$\int_A yz\, F_3(x)\,dA = 0 \tag{10-d}$$

and

$$\int_A \omega_p(z,y) f_4(x)\,dA = 0 \tag{11-a}$$

$$\int_A y\, \omega_p(z,y) f_5(x)\,dA = 0 \tag{11-b}$$

$$\int_A z\, \omega_p(z,y) f_6(x)\,dA = 0 \tag{11-c}$$

Equations (10) are satisfied because Y and Z are principal centroidal axes, while Equations (11) are satisfied because the warping axial stresses are self equilibrating over the entire cross-section.

Also define

$$\int_A dA = A \tag{12-a}$$

$$\int_A z^2\, dA = I_y \tag{12-b}$$

$$\int_A y^2\, dA = I_z \tag{12-c}$$

$$\int_A \omega_p^2(y,z)dA = I_\Omega \qquad (12\text{-}d)$$

(9) now reads

$$U_s = \frac{E}{2}\int_0^{\ell} \{A\, U_{o,x}^2 + I_z\, V_{s,xx}^2 + I_y\, W_{s,xx}^2 + I_\Omega\, \beta_{,xx}^2\}\, dx$$

$$+ \frac{G}{2}\int_0^{\ell} J\, \beta_{,x}^2\, dx \qquad (13)$$

The work done by end forces acting at element terminals is given as,

$$W = P_p\, U_{o_p} + P_q\, U_{o_q} + M_{\lambda_p}\, \lambda_p + M_{\lambda_q}\, \lambda_q$$

$$+ M_{\theta_p}\, \theta_p + M_{\theta_q}\, \theta_q + M_{\beta_p}\, \beta_p + M_{\beta_q}\, \beta_q$$

$$+ M_{\Omega_p}\, \beta_{,x_p} + M_{\Omega_q}\, \beta_{,x_q} + Q_{v_p}\, V_{s_p} + Q_{v_p}\, V_{s_q}$$

$$+ Q_{w_p}\, W_{s_p} + Q_{w_q}\, W_{s_q} \qquad (14)$$

It may be noted from (1) that the axial displacement U_{o_p} may not be zero at the centroid ($y = z = 0$) because of the warping stresses. However, due to the mono-symmetry of the rail cross-section, $\omega_p(0,0) = 0$, hence

U_{o_p} and U_{o_q} are the only non-zero components at the centroid ($y = z = 0$). It will also be shown later, Equations (39,40), that

$$\lambda = V_{s,x} \quad \text{(at the shear centre)} \tag{15}$$

$$\theta = -W_{s,x} \quad \text{(at the shear centre)} \tag{16}$$

where V_s and W_s are the displacements of the shear centre along Y and Z axes, respectively. The total potential energy for the element is defined as

$$\pi_p = U_s - W \tag{17}$$

and for equilibrium

$$\delta\pi_p = 0 \tag{18}$$

which yields the following set of governing differential equations and boundary conditions.

1.1 Governing Differential Equations

$$EA\, U_{o,xx} = 0 \tag{19}$$

$$EI_z\, V_{s,xxxx} = 0 \tag{20}$$

$$EI_y\, W_{s,xxxx} = 0 \tag{21}$$

$$EI_{\Omega}\,\beta_{,xxxx} - GJ\,\beta_{,xx} = 0 \tag{22}$$

1.2 Boundary Conditions

$$[EAU_{o_{,x}} - P]\,\delta\,U_{o_q} = 0 \tag{23}$$

$$[EAU_{o_{,x}} + P]\,\delta\,U_{o_p} = 0 \tag{24}$$

$$[EI_z\,V_{s_{,xx}} - M_{\lambda}]\,\delta\,V^{q}_{s_{,x}} = 0 \tag{25}$$

$$[EI_z\,V_{s_{,xx}} + M_{\lambda}]\,\delta\,V^{p}_{s_{,x}} = 0 \tag{26}$$

$$[EI_y\,W_{s_{,xx}} + M_{\theta}]\,\delta\,W^{q}_{s_{,x}} = 0 \tag{27}$$

$$[EI_y\,W_{s_{,xx}} - M_{\theta}]\,\delta\,W^{p}_{s_{,x}} = 0 \tag{28}$$

$$[EI_{\Omega}\,\beta_{,xx} - M_{\Omega}]\,\delta\,\beta^{q}_{,x} = 0 \tag{29}$$

$$[EI_{\Omega}\,\beta_{,xx} + M_{\Omega}]\,\delta\,\beta^{p}_{,x} = 0 \tag{30}$$

$$[EI_z\,V_{s_{,xxx}} + Q_v]\,\delta\,V_{s_q} = 0 \tag{31}$$

$$[EI_z\,V_{s_{,xxx}} - Q_v]\,\delta\,V_{s_p} = 0 \tag{32}$$

$$[EI_y\,W_{s_{,xxx}} + Q_w]\,\delta\,W_{s_q} = 0 \tag{33}$$

$$[EI_y\,W_{s_{,xxx}} - Q_w]\,\delta\,W_{s_p} = 0 \tag{34}$$

$$[(GJ\ \beta_{,x} - EI_{\Omega}\ \beta_{,xxx}) - M_{\beta}]\ \delta\ \beta_{q} = 0 \tag{35}$$

$$[(GJ\ \beta_{,x} - EI_{\Omega}\ \beta_{,xxx}) + M_{\beta}]\ \delta\ \beta_{p} = 0 \tag{36}$$

2. Stability Criterion

A suitable stability criterion normally adopted for elastic structures with conservative loading is a positive definite second variation of the total potential energy of the structure. A convenient expression for the second variation of the total potential energy based on a Lagrangian coordinate system is used in the following. The expression is written in terms of the perturbing strains and the initial equilibrium stress state.

$$\delta^2(\pi_p) = \int_V \{\tfrac{1}{2}\ (\overset{(1)}{\delta\varepsilon_{ij}})\ E^{ij}_{pq}\ (\overset{(1)}{\delta\varepsilon_{pq}}) + \overset{(0)}{S_{ij}}\ (\overset{(2)}{\delta\varepsilon_{ij}})\}dV \tag{37}$$

where E^{ij}_{pq} refer to the appropriate elastic constants

$\overset{(0)}{S_{ij}}$ is the Kirchoff stress in the initial equilibrium state prior to any kinematically admissible perturbation of the displacement field. For small initial deformations, in the first order approximation, the Kirchoff stress tensor $\overset{(0)}{S_{ij}}$ is equal to the Eulerian stress tensor σ_{ij}.

$$\overset{(1)}{\delta\varepsilon_{ij}} = \tfrac{1}{2}\ (\delta U_{i,j} + \delta U_{j,i})$$

$$\delta\varepsilon_{ij}^{(2)} = \frac{1}{2}\left(\delta U_{k,i} \quad \delta U_{k,j}\right)$$

where (δU_i) is the displacement perturbation component in the (i) direction.

The relevant strain and stress components for rail analysis are

a) ε_{xx} , σ_{xx} due to axial, flexural and warping movements

b) γ_{xy} , τ_{xy} due to flexure and torsion

c) γ_{xz} , τ_{xz} due to flexure and torsion.

It has to be mentioned that, in the theoretical buckling problem where initial imperfections are not considered, the equilibrium position at buckling is essentially different from the initial prebuckling configuration. In the presence of bimoment and/or biaxial bending, the stability loss is not associated with the setting in of a new equilibrium position but is marked by excessive increase in some (or all) of the prebuckling deformation components.

The first component on the R.H.S. of the integral in (37) represents the linear strain energy in the perturbed position. Both the axial strain energy and St. Venant torsional strain energy will be considered in evaluating this component of the integral in (37) for reasons discussed earlier in this section. The second component of the integral in (37) reflects the effect of the perturbing stress on the

nonlinear strain component associated with the same stress components.

From the previous discussions, Equation (37) reads

$$\delta^2(\pi_p) = \frac{E}{2}\int_V (\delta\varepsilon_{xx}^{(1)})(\delta\varepsilon_{xx}^{(1)})\,dV + \frac{G}{2}\sum_i \int_{V_i} (\delta\gamma_i)^2\,dv_i$$

$$+ \int_V \sigma_{xx}^{(0)}\,(\delta\varepsilon_{xx}^{(2)})\,dV$$

$$+ 2\int_V \tau_{xy}^{(0)}\,(\delta\varepsilon_{xy}^{(2)})\,dV$$

$$+ 2\int_V \tau_{xz}^{(0)}\,(\delta\varepsilon_{xz}^{(2)})\,dV \qquad (38)$$

2.1 Lateral-Torsional Buckling Displacements

Complementary to Equation (1), the displacement of any point (x,y,z) along the Y and Z axes are

$$v(x,y,z) = V_s - z\,\beta(x) \qquad (39)$$

$$w(x,y,z) = W_s + (y+e_y)\,\beta(x) \qquad (40)$$

Equations (39,40) imply that the cross-section rotates rigidly about the shear centre and retains its shape in the pre and post buckling

configurations. e_y represents the Y coordinate of the shear centre and is positive, as shown in Figure 1 of Part III. Equations (39,40) are valid for total displacements and variation of displacements.

In case of linear static problems, the solution presented is exact since the displacement modes are the solution of the governing differential equations. For stability problems the situation is different since the choice of displacement modes is a matter of assumption. It is assumed, in this study, that the static deformation modes are similar in shape to the buckling modes. This assumption is frequently and successfully used in conjunction with variational methods. It is, therefore, expected that in stability analysis, the solution provides a close upper bound to the buckling load and the accuracy becomes dependent on the number of elements employed. It can be easily shown that the first two integrals in (38) yield the following result as indicated in the section on variational analysis

$$= \int_{\ell} [\tfrac{1}{2} EA(\delta U_{o,x})^2 + \tfrac{1}{2} EI_z (\delta V_{x,xx})^2 + \tfrac{1}{2} EI_y (\delta W_{s,xx})^2$$

$$+ \tfrac{1}{2} E I_{\Omega} (\delta\beta_{,xx})^2 + \tfrac{1}{2} GJ (\delta\beta_{,x})^2] \, dx \qquad (41)$$

which is identical to (13) except that total displacements are replaced by variation. Calculating the other terms is now in order.

2.2 Evaluating $I_3 = \int_V \overset{(0)}{\sigma}_{xx} \overset{(2)}{(\delta\varepsilon_{xx})}\, dV$ in Equation (38)

$$\overset{(2)}{\delta\varepsilon}_{xx} = \tfrac{1}{2}\left[(\delta U_{,x})^2 + (\delta V_{,x})^2 + (\delta W_{,x})^2\right] \tag{42}$$

For long prismatic beams $(\delta U_{,x})^2 << (\delta V_{,x})^2$ or $(\delta W_{,x})^2$ and therefore

$$\overset{(2)}{\delta\varepsilon}_{xx} = \tfrac{1}{2}\left[(\delta V_{,x})^2 + (\delta W_{,x})^2\right] \tag{43}$$

From (39) and (40)

$$\overset{(2)}{\delta\varepsilon}_{xx} = \tfrac{1}{2}\left[(\delta V_{s,x} - z\,\delta\beta_{,x})^2 + (\delta W_{s,x} + (y+e_y)\delta\beta_{,x})^2\right] \tag{44}$$

The third integral in (38) now reads

$$I_3 = \int_V \frac{\sigma_{xx}}{2} \{(\delta V_{s,x})^2 + z^2\,(\delta\beta_{,x})^2 - 2z\,(\delta V_{s,x}\,\delta\beta_{,x})$$

$$+ (\delta W_{s,x})^2 + (y+e_y)^2(\delta\beta_{,x})^2 + 2(y+e_y)(\delta W_{s,x}\,\delta\beta_{,x})\}\, dV$$

Regrouping terms we get

$$I_3 = \int_V \frac{\sigma_{xx}}{2} \{(\delta V_{s,x})^2 + (\delta W_{s,x})^2 + (z^2+(y+e_y)^2)(\delta\beta_{,x})^2$$

$$+ (-2z)(\delta V_{s,x}\,\delta\beta_{,x}) + 2(y+e_y)(\delta W_{s,x}\,\delta\beta_{,x})\}\, dV \tag{45}$$

Substituting the expression for $\overset{(0)}{\sigma}_{xx}$ from Equation (12) in Part III into (45) and using the section properties in (10-11 & 12) the following results

$$I_3 = \frac{1}{2}\left[\int_{\ell} (\delta V_{s,x})^2 \cdot N(x) + (\delta W_{s,x})^2 \cdot N(x)\right.$$

$$+ (\delta\beta_{,x})^2 \left\{N(x)\frac{I_{p_o}}{A} + M_\lambda(x)\frac{\beta_1}{I_z} + M_\theta(x)\frac{\overset{0}{\beta_2}}{I_y} + M_\Omega(x)\frac{\overset{0}{\beta_3}}{I_\Omega}\right.$$

$$+ (\delta V_{s,x}\ \delta\beta_{,x})(2\ M_\theta(x))$$

$$\left. + (\delta W_{s,x}\ \delta\beta_{,x})(-2M_\lambda(x)+2N(x)\cdot e_y)dx\right] \qquad (46)$$

The following constants are now defined

$$I_{p_o} = \int_A z^2 + (y+e_y)^2\ dA \qquad (47)$$

$$\beta_1 = \int_A -y(z^2+y^2+2ye_y)\ dA \qquad (48)$$

$$\beta_2 = \int_A -zy^2\ dA$$

$$\beta_3 = \int_A \omega_p(y,z)(z^2+y^2)\ dA$$

Due to the mono-symmetry of the rail cross-section about Y-axis β_2 & β_3 are identically zero.

2.3 Evaluating $I_4 = \int_V 2\, \overset{(0)}{\tau_{xy}}\, (\overset{(2)}{\delta\varepsilon_{xy}})\, dV$ in Equation (38)

$\overset{(0)}{\tau_{xy}}$ results generally from (a) Direct shear force Q_v

(b) Non-uniform biomoment

(c) St. Venant torsion

In the case of the rail section, the bimoment is assumed to be taken by the rail head and rail flange which, because of their orientation, only have $\overset{(0)}{\tau_{xz}}$; therefore non-uniform bimoment has no effect on I_4. St. Venant torsion introduces $\overset{(0)}{\tau_{xy}}$ in the web. It is shown, however, in Appendix C that St. Venant torsion has no effect on the lateral-torsional stability of the rail section. Therefore $\overset{(0)}{\tau_{xy}}$ due to St. Venant torsion will be neglected. The only remaining effect is $\overset{(0)}{\tau_{xy}}$ due to direct shear force, $Q_v(x)$, and is now discussed.

According to beam theory

$$\overset{(0)}{\tau_{xy}} = \frac{-Q_v \cdot S_z(y)}{I_z \cdot t_p(y)} \tag{49}$$

where

$$S_z(y) = \int_o^y y\, dA \tag{50}$$

$$Q_v = \frac{d\, M_\lambda(x)}{dx} = Q_{v_p}, \tag{51}$$

Also

$$\overset{(2)}{\delta\varepsilon}_{xy} = \frac{1}{2}\,(\delta W_{,x}\ \delta W_{,y}) + \text{small terms}$$

$$= \frac{1}{2}\,\{(\delta W_{s,x} + (y+e_y)\ \delta\beta_{,x})(\delta\beta)\}$$

$$= \frac{1}{2}\,\{\delta W_{s,x}\ \delta\beta + (y+e_y)\ \delta\beta_{,x}\ \delta\beta\} \tag{52}$$

$$I_4 = \int_V 2 \quad \frac{1}{2}\ \overset{(0)}{\tau}_{xy}\ (\delta W_{s,x}\ \delta\beta + (y+e_y)\ \delta\beta_{,x}\ \delta\beta)\ dV$$

$$= \int_0^{\ell} \delta W_{s,x}\ \delta\beta \int_A \overset{(0)}{\tau}_{xy}\ dA + \int_0^{\ell} \delta\beta_{,x}\ \delta\beta \int_A (y+e_y)\ \overset{(0)}{\tau}_{xy}\ dA \tag{53}$$

using (49) in (53) and realizing that

$$\int_A \overset{(0)}{\tau}_{xy}\ dA = -\ Q_v \tag{54}$$

we get

$$I_4 = -\ Q_v\ \{\int_0^{\ell} \delta W_{s,x}\ \delta\beta\ dx + \frac{\gamma}{I_z} \int_0^{\ell} \delta\beta\ \delta\beta_{,x}\ dx\} \tag{55}$$

where

$$\gamma = \int_A \frac{(y+e_y)\quad S_z(y)}{t_p(y)}\ dA \tag{56}$$

$$S_z(y) = \int_A y \, dA$$

2.4 Evaluating $I_5 = \int_V 2 \, \overset{(0)}{\tau_{xy}} \, (\overset{(2)}{\delta\varepsilon_{xz}}) \, dV$ in Equation (38)

$\overset{(0)}{\tau_{xz}}$ results from (a) Direct shear Q_w

(b) Non-uniform bimoment

(c) St. Venant torsion.

The contribution of St. Venant torsion on lateral-torsional buckling is shown to be zero in Appendix C. The other two effects are now discussed.

2.4.1 Due to Q_v

$$\tau_{xz} = \frac{-Q_v \cdot S_y(z)}{I_y \cdot t_p(z)} \tag{57}$$

where $$S_y(z) = \int_o^z z \, dA \tag{58}$$

$$Q_v = \frac{dM_\theta(x)}{dx} = Q_{w_p} \tag{59}$$

2.4.2 Due to M_Ω

Non-uniform bimoment causes shear forces in the top and bottom

flanges of equal and opposite magnitude. It will be shown later in this section that the contribution of non-uniform bimoment to I_5 is identically zero.

Since

$$\overset{(2)}{\delta\varepsilon_{xz}} = \frac{1}{2}\,(\delta V_{,x}\ \delta V_{,z}) + \text{small terms}$$

$$= \frac{1}{2}\,\{(\delta V_{s_{,x}} - z\delta\beta_{,x})(-\delta\beta) \tag{60}$$

Therefore

$$I_5 = \int_V \overset{(0)}{\tau_{xz}}\,(-\delta V_{s_{,x}}\ \delta\beta + z\ \delta\beta_{,x}\ \delta\beta) \tag{61}$$

$\overset{(0)}{\tau_{xz}}$ is now split into $\overset{(0)}{\tau_{xz_s}}$ and $\overset{(0)}{\tau_{xz_b}}$ where

$\overset{(0)}{\tau_{xz_s}}$ is the shearing stress due to Q_w and calculated according to (57). $\overset{(0)}{\tau_{xz_b}}$ is the shear stress due to the non-uniform bimoment effect.

$$I_5 = \int_V \overset{(0)}{\tau_{xz_s}}\,(-\delta V_{s_{,x}}\ \delta\beta + z\ \delta\beta_{,x}\ \delta\beta)\ dV$$

$$+ \int_V \overset{(0)}{\tau_{xz_b}}\,(-\delta V_{s_{,x}}\ \delta\beta + z\delta\beta_{,x}\ \delta\beta)\ dV \tag{62}$$

$$= \left[\int_0^{\ell} -\delta V_{s,x}\, \delta\beta\, dx \int_A {}^{(0)}\tau_{xz_s}\, dA\right] + \left[\int_0^{\ell} \delta\beta_{,x}\, \delta\beta\, dx \int_A {}^{(0)}\tau_{xz_s}\, z\, dA\right]$$

$$+ \left[\int_0^{\ell} -\delta V_{s,x}\, \delta\beta\, dx \int_A {}^{(0)}\tau_{xz_b}\, dA\right] + \left[\int_0^{\ell} \delta\beta_{,x}\, \delta\beta\, dx \int_A {}^{(0)}\tau_{xz_b}\, z\, dA\right] \quad (63)$$

$$= I_{51} + I_{52} + I_{53} + I_{54} \quad (64)$$

$$I_{51} = Q_w \int_0^{\ell} \delta V_{s,x}\, \delta\beta\, dx \quad (65)$$

$I_{52} = 0.$ (since shear stresses are symmetrical about Y axis)

$I_{53} = 0.$ (since the total shear force in the top and bottom flanges are equal and opposite due to non-uniform bimoment)

$I_{53} = 0.$ (since shear stresses due to non-uniform bimoment are symmetrical about Y axis).

3. Rail Element Geometric Stiffness Matrix

Equation (38), after incorporating Equations (41,46,55 & 65) can be put in a matrix form as

$$\overset{2}{\delta}(\pi_p) = \frac{1}{2}\int_{\ell} \{\delta\alpha\}^T [\bar{k}] \{\delta\alpha\} + \{\delta\rho\}^T [\bar{S}] \{\delta\rho\}\, dx \tag{66}$$

$$= \frac{1}{2}(R_1 + R_2)$$

where

$$\{\delta\alpha\}^T = \{\delta U_{o,x} \quad \delta V_{s,xx} \quad \delta W_{s,xx} \quad \delta\beta_{,xx} \quad \delta\beta_{,x}\} \tag{67}$$

$$\{\delta\rho\}^T = \{\delta V_{s,x} \quad \delta W_{s,x} \quad \delta\beta_{,x} \quad \delta\beta\} \tag{68}$$

$$[\bar{k}] = \lceil EA \quad EI_z \quad EI_y \quad EI_\Omega \quad GJ \rfloor \tag{69}$$

where the special bracket $\lceil\ \rfloor$ indicates a diagonal matrix.

$$[\bar{S}] = \begin{bmatrix} N(x) & 0 & M_\theta(x) & Q_w(x) \\ 0 & N(x) & -M_\lambda(x)+e_y\cdot N(x) & -Q_v(x) \\ M_\theta(x) & -M_\lambda(x)+e_y\cdot N(x) & \psi & -Q_v(x)\cdot\eta \\ Q_w(x) & -Q_v(x) & -Q_v(x)\cdot\eta & 0 \end{bmatrix}$$

where

$$\psi = N_x \frac{I_{p_o}}{A} + M_\lambda(x) \frac{\beta_1}{I_z} \tag{71}$$

$$\eta = \frac{\gamma}{I_z} \tag{72}$$

Following a series of transformations, the matrix (69) will eventually yield the linear elastic stiffness matrix shown in Appendix B. The steps to obtain the element geometric stiffness matrix are now given.

First we need to obtain the displacement functions for the continuous displacements $\delta U_o(x)$, $\delta V_s(x)$, $\delta W_s(x)$ and $\delta\beta(x)$. This is possible by finding the solution to the governing differential Equations (19-20) for the displacements which are assumed to have the same form as their variations.

$$\delta U_o(x) = \delta C_1\, x + \delta C_8 \tag{73}$$

$$\delta V_s(x) = \delta C_2\, x^3 + \delta C_7\, x^2 + \delta C_9\, x + \delta C_{14} \tag{74}$$

$$\delta W_s = \delta C_3\, x^3 + \delta C_6\, x^2\, \delta C_{10}\, x + \delta C_{13} \tag{75}$$

$$\delta\beta(x) = \delta C_4\, \cosh\kappa x + \delta C_5\, \sinh\kappa x + \delta C_{11}\, \kappa x + \delta C_{12} \tag{76}$$

It now becomes possible to express $\{\delta\rho\}$ in (68) as

$$\{\delta\rho\} = [B_s]\,\{\delta C\} \tag{77}$$

where

$$[B_s] = \begin{bmatrix} 0 & 3x^2 & 0 & 0 & 0 & 0 & 2x & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3x^2 & 0 & 0 & 2x & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{a} & \bar{b} & 0 & 0 & 0 & 0 & 0 & \bar{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & ch & sh & 0 & 0 & 0 & 0 & 0 & \kappa x & 1 & 0 & 0 \end{bmatrix} \quad (78)$$

$$\{\delta C\}^T = \{\delta C_1, \delta C_2, \delta C_3, \delta C_4, \delta C_5, \delta C_6, \delta C_7, \delta C_8, \delta C_9, \delta C_{10}, \delta C_{11}, \delta C_{12}, \delta C_{13}, \delta C_{14}\} \quad \ldots (79)$$

in (78) $\bar{a} = \kappa \sinh \kappa x$

$\bar{b} = \kappa \cosh \kappa x$

$\bar{c} = \kappa$

$ch = \cosh \kappa x$

$sh = \sinh \kappa x$

where $\kappa = (GJ/EI_\Omega)^{1/2}$

The vector of polynomial coefficients $\{\delta C\}$ is related to the vector of element nodal displacements through the relationship

$$\{\delta C\} = [C]^{-1} \{\delta q_e\} \quad (80)$$

where

$$\{\delta q_e\}^T = \{U_{op}, V_{s_p}, W_{s_p}, \beta_p, \beta_p', \theta_p, \lambda_p \ , \ U_{oq}, V_{sq}, \beta_q, \beta_q', \theta_q, \lambda_q\} \quad (81)$$

The transformation matrix $[C]^{-1}$ is given in Appendix D

$\{\delta\rho\}$ in (77) can now be expressed as

$$\{\delta\rho\} = [B_s]\,[C]^{-1}\,\{\delta q_e\} \tag{82}$$

Substituting (82) into the second integral, R_2, in Equation (66) yields

$$R_2 = \tfrac{1}{2}\,\{\delta q_e\}^T\,[[C]^{-1}]^T \int_\ell [B_s]^T\,[\bar{S}]\,[B_s]\,dx\,[C]^{-1}\,\{\delta q_e\} \tag{83}$$

$$= \tfrac{1}{2}\,\{\delta q_e\}^T\,[k_G]\,\{\delta q_e\} \tag{84}$$

where $[k_G]$ is the element geometric stiffness matrix defined as

$$[k_G] = [[C^{-1}]]^T \int_0^\ell [B_s]^T\,[\bar{S}]\,[B_s]\,dx\,[C]^{-1} \tag{85}$$

4. Special Problems Associated With Rail Stability

4.1 Second Variation of Nodal Transverse Loads and Reactions Eccentric to the Shear Centre

The transverse displacements v and w in (39,40) are linear combinations of the shear centre displacements V_s, W_s and β. By considering the rotation of a point about the shear centre and assuming large displacements, higher order expressions for v and w are obtained as

$$v(x,y,z) = V_s(x) - z\beta - \tfrac{1}{2}\,(y+e_y)\beta^2 \tag{86}$$

$$w(x,y,z) = W_s(x) + (y+e_y)\beta - \frac{1}{2} z \beta^2 \qquad (87)$$

The second order terms vanish at the shear centre. For consistency in determining a load potential when transverse loads eccentric to the shear centre are applied, these second order effects cannot be ignored. Equations (86,87) can also be expressed in the form of displacement variations.

Since only second order variations are significant in the stability criterion employed, the second order variation of the external load potential is given by

$$\delta^2(W) = \frac{1}{2} \sum_J \{P_y^m (-a_y^m) + P_z^m (-a_z^m)\} (\delta\beta_m)^2 \qquad (88)$$

($a_z^m \simeq 0.0$)

where a_y^m and a_z^m are the eccentricities of load components P_y and P_z acting at node m w.r.t. the shear centre. J is the total number of loaded nodes. Notice that a_z^m is equal to one-half the web thickness which is small and may be taken as zero.

The same argument can be extended to the eccentricity of the foundation vertical spring at each node w.r.t. the shear centre. The result is an extra term to be added to the second variation of the structural total potential energy and takes the following form

$$\delta^2(U_s) = \frac{1}{2} \sum_j F_v^m (-d^m+e_y) (\delta\beta_m)^2 \qquad (89)$$

F_v^m is the force in the Y direction in the spring with stiffness k_v at

node m. d^m is the "y" coordinate of the rail base where the vertical spring is assumed to act. Positive d and e_y are indicated in Figure 1.

5. Total Second Variation of the Structural Potential Energy, $\delta^2(\Pi p)$

This quantity can be expressed as

$$\delta^2 (\Pi p) = \frac{1}{2} \{\delta \underline{q}^e\}^T [K_e] \{\delta \underline{q}^e\} + \frac{1}{2} \{\delta \underline{q}^e\} [K_G] \{\delta \underline{q}^e\} \qquad (90)$$

where $\{\delta \underline{q}^e\}$ is the vector of variation of the global nodal displacements

$[K_e]$ is the structural global elastic stiffness matrix representing the linear stiffness of the rail and spring elements.

$[K_G]$ is the structural geometric stiffness matrix defined as

$$\sum_i [k_G] - \sum_{j_\beta} p_y^m (-a_y^m) + \sum_{j_\beta} F_v^m (-d^m + e_y) \qquad (91)$$

where i is the number of rail elements. j_β represents the total torsional rotation degrees of freedom (β) of the entire structure. The summation in (91) is vectorial.

Equation (90) can be re-cast as

$$\delta^2 (\Pi_p) = \frac{1}{2} \{\delta \underline{q}^e\}^T [K_e + K_G] \{\delta \underline{q}^e\} \qquad (92)$$

For $\delta^2 (\Pi p)$ to be positive for any $\{\delta \underline{q}^e\} \neq 0$ the matrix $[K_e + K_G]$ must be positive definite. If the external loads are proportional, then (92)

can be solved as an eigenvalue problem. To give more flexibility to the analysis in the event of general non-proportional external loading the smallest eigenvalue of the $[K_e + K_G]$ matrix will be evaluated at the outset of each load increment. The stress state that corresponds to the vanishing of the smallest eigenvalue will correspond to bifurcation of the rail. The stability criterion can be numerically expressed as

$$\det [K_e + K_G] > 0 \quad \text{the rail is stable} \tag{93}$$

$$\det [K_e + K_G] < 0 \quad \text{the rail is unstable} \tag{94}$$

$$\text{But, since, } \det [K_e + K_G] = \lambda_1 \cdot \lambda_2 \cdot \cdot \cdot \lambda_n \tag{95}$$

the vanishing of λ_1 (smallest eigenvalue) is a necessary and sufficient condition for stability. The stability criterion employed in the current program is then

$$\text{If } \lambda_{min} > 0 \quad \text{rail is stable} \tag{96}$$

$$\text{If } \lambda_{min} < 0 \quad \text{rail is unstable} \tag{97}$$

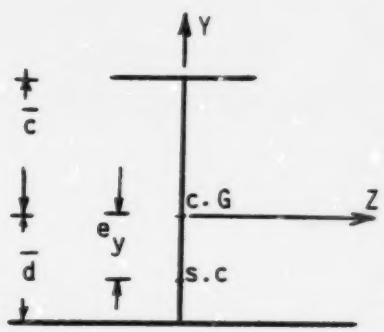
The critical loads correspond to the case when

$$\lambda_{min} = 0 \tag{98}$$

Appendix E contains listings of the program used in the analysis.

6. Confirmatory Examples

An unequal flanged I-section is used in the following examples (smaller top flange). The cross-sectional and material properties are listed below



$A = 0.3\ (10)^3\ mm^2$

$I_z = 0.245\ (10)^6\ mm^4$

$I_y = 0.927\ (10)^4\ mm^4$

$J = 0.7621\ (10)^3\ mm^4$

$I_\Omega = 0.486\ (10)^7\ mm^6$

$I_{p_o} = 0.4007\ (10)^6\ mm^4$ [in Equation (47)]

$\beta_1 = -0.12697\ (10)^8\ mm^5$ [in Equation (48)]

$\gamma = 0.6450\ (10)^7\ mm^5$ [in Equation (56)]

$e_y = 0.2210\ (10)^2\ mm$

$\bar{c} = 0.4213\ (10)^2\ mm$

$\bar{d} = 0.3031\ (10)^2$ mm

$E = 0.6512\ (10)^5$ MPa

$G = 0.2597\ (10)^5$ MPa

6.1 Simple Beam Under Axial Thrust

A simple beam of total length equal to 2540 mm is subject to axial compressive forces at both ends acting through the centroid. The analytical solution gives a value for N_{cr} = 902.0 N. The present finite element solution predicts N_{cr} = 901.8 N. Figure 1 shows the pre buckling displacements along beam centroid, while Figure 2 shows the relative buckling displacements along the rail shear centre. By comparing the two figures it is seen that an interactive lateral-torsional buckling mode occurs.

If the lateral degrees of freedom are suppressed and the axial compression is applied through the shear centre instead of the centroid, the analytical solution gives N_{cr} = 102.3 KN. The present finite element solution gives N_{cr} = 101.6 KN. The buckling mode in this case is pure torsion. Figures 3 and 4 show the centroidal displacements and the relative shear centre displacements in pre and post buckling, respectively.

6.2 Simple Beam Under End Moments

A simply supported beam of length 6000 mm is subject to terminal moments causing tension in the top (smaller) flange. The analytical

Displacements Along Beam Centroid

(in mms ,radians and radians per mm)

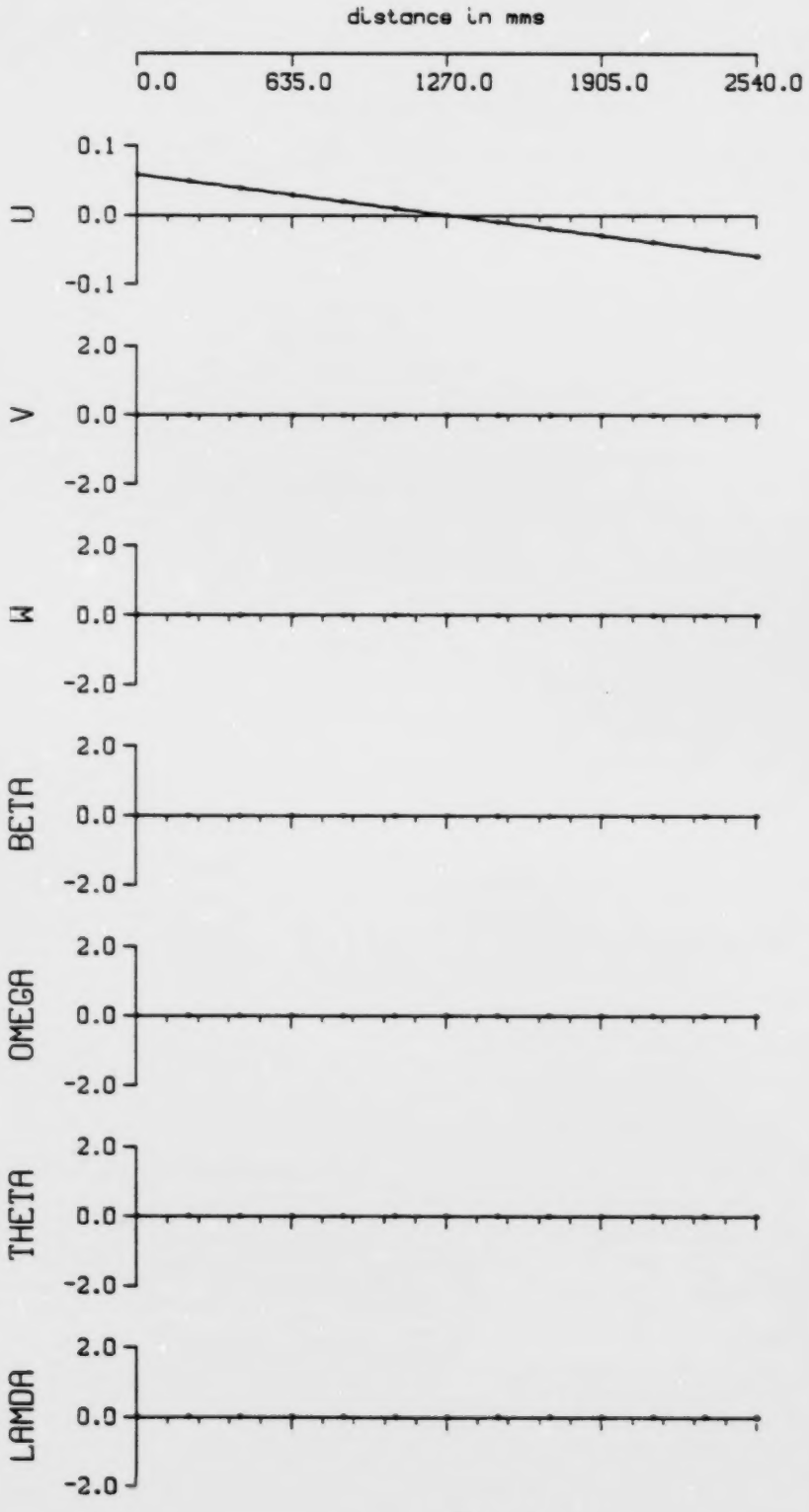


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Fig. 1

Buckling Modes Along Beam Shear Centre
(Dimensionless)

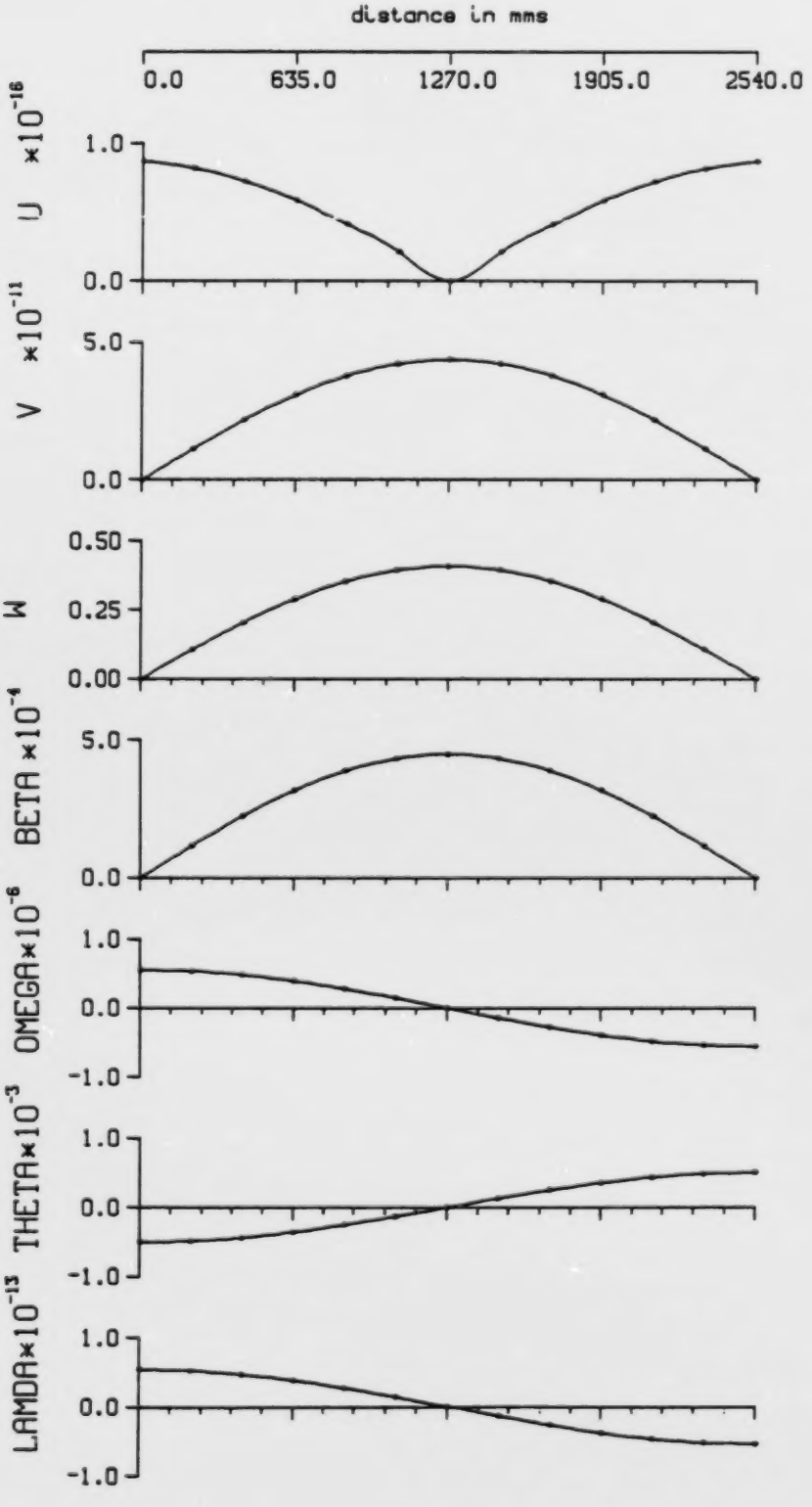

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Fig. 2

Displacements Along Beam Centroid

(in mms ,radians and radians per mm)

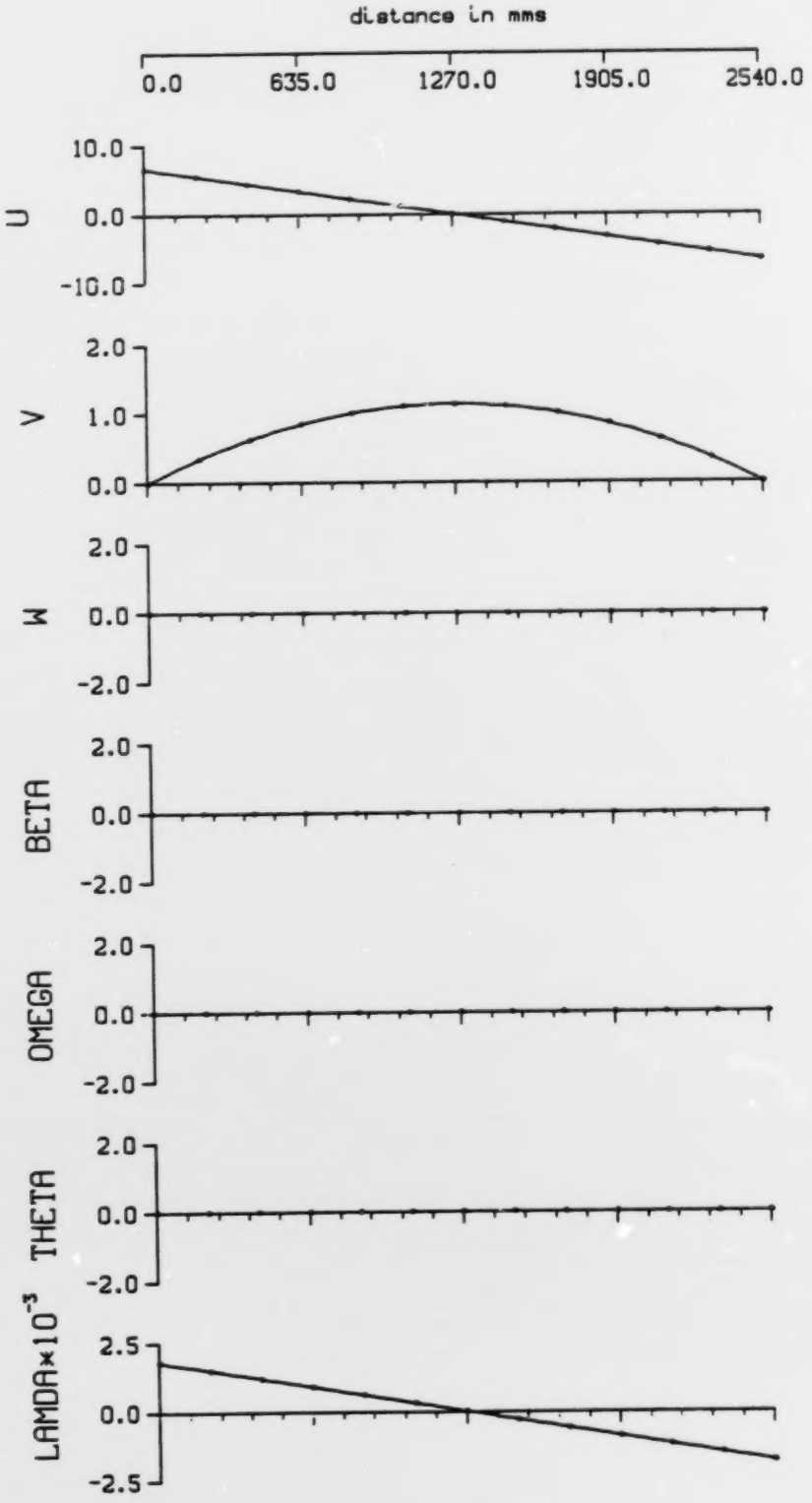

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Fig. 3

Buckling Modes Along Beam Shear Centre
(Dimensionless)

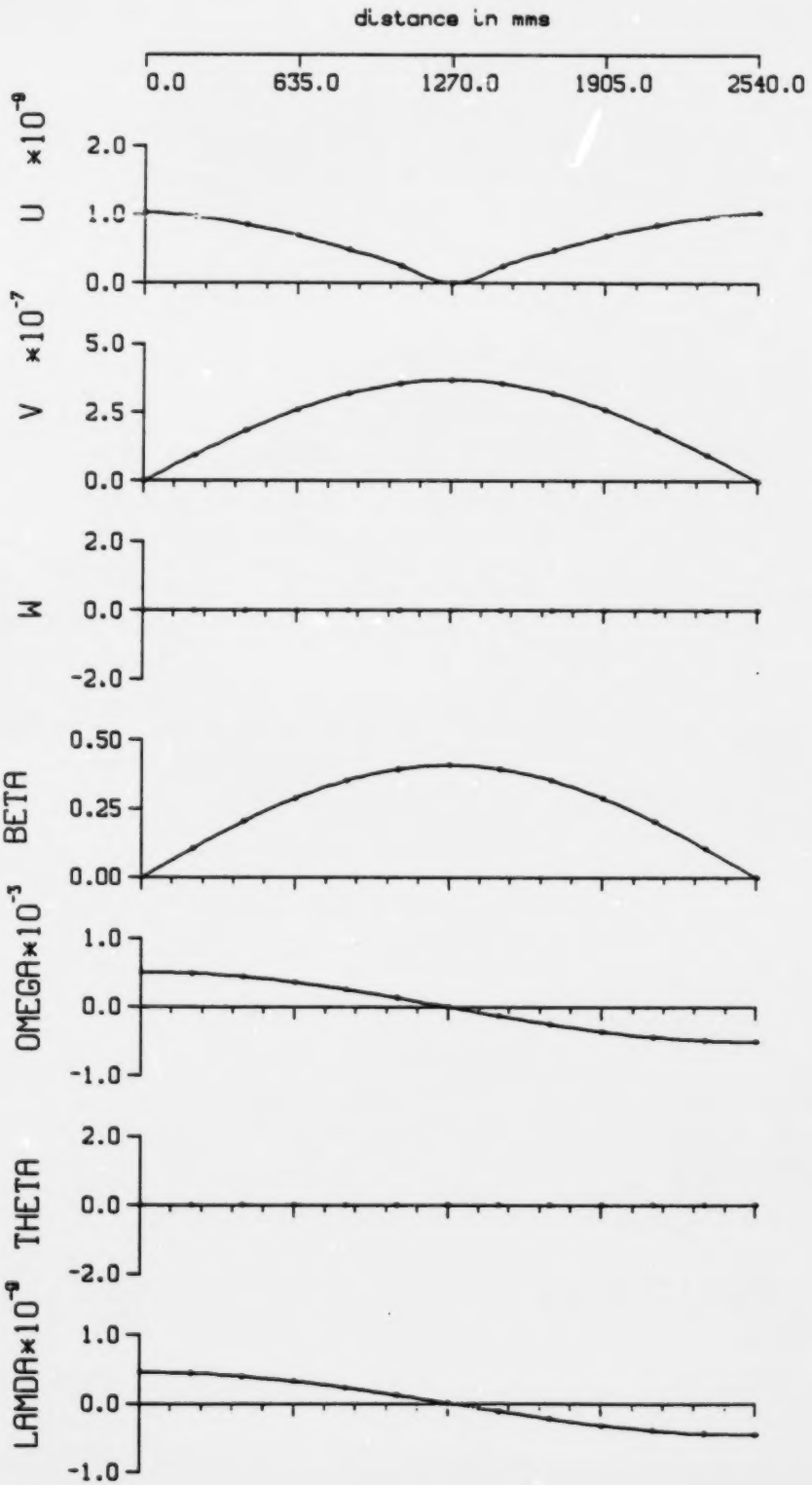

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Fig. 4

solution gives M_{cr} = 60.45 KN.mm, while the present finite element solution gives M_{cr} = 60.5 KN.mm. The buckling mode is lateral-torsional buckling. Figures 5,6 show the centroidal pre-buckling displacements and the shear centre post-buckling relative displacements, respectively.

6.3 Cantilever Beam Under End Load

A cantilever beam of length 1270 mm is used in this example. The beam is subject to increasing downwards vertical load applied on the top flange at eccentricity equal to 65.8 mm from the shear centre. A 3-D composite finite element solution gives P_{cr} = 192 N, the present finite element solution gives P_{cr} = 194 N. Figures 7,8 show the centroidal prebuckling displacement and the shear centre post buckling relative displacement, respectively. It is interesting to note that, if the vertical load is applied through the bottom flange, at an eccentricity of -9.77 mm to the shear centre, the buckling load increases to P_{cr} = 413.5 N. The 3-D composite solution gives P_{cr} = 410 N.

The results obtained do not consistently confirm that an upper bound to the analytical buckling load is obtained. This can be justified by realizing that the numerical technique employed brackets the buckling load between two close values where, at the lower value, the structure is stable while at the other value the structure is unstable. The finite element values given in the previous examples can be either of these two values depending on the tightness of the exit criterion used to terminate the analysis.

Displacements Along Beam Centroid

(in mms ,radians and radians per mm)

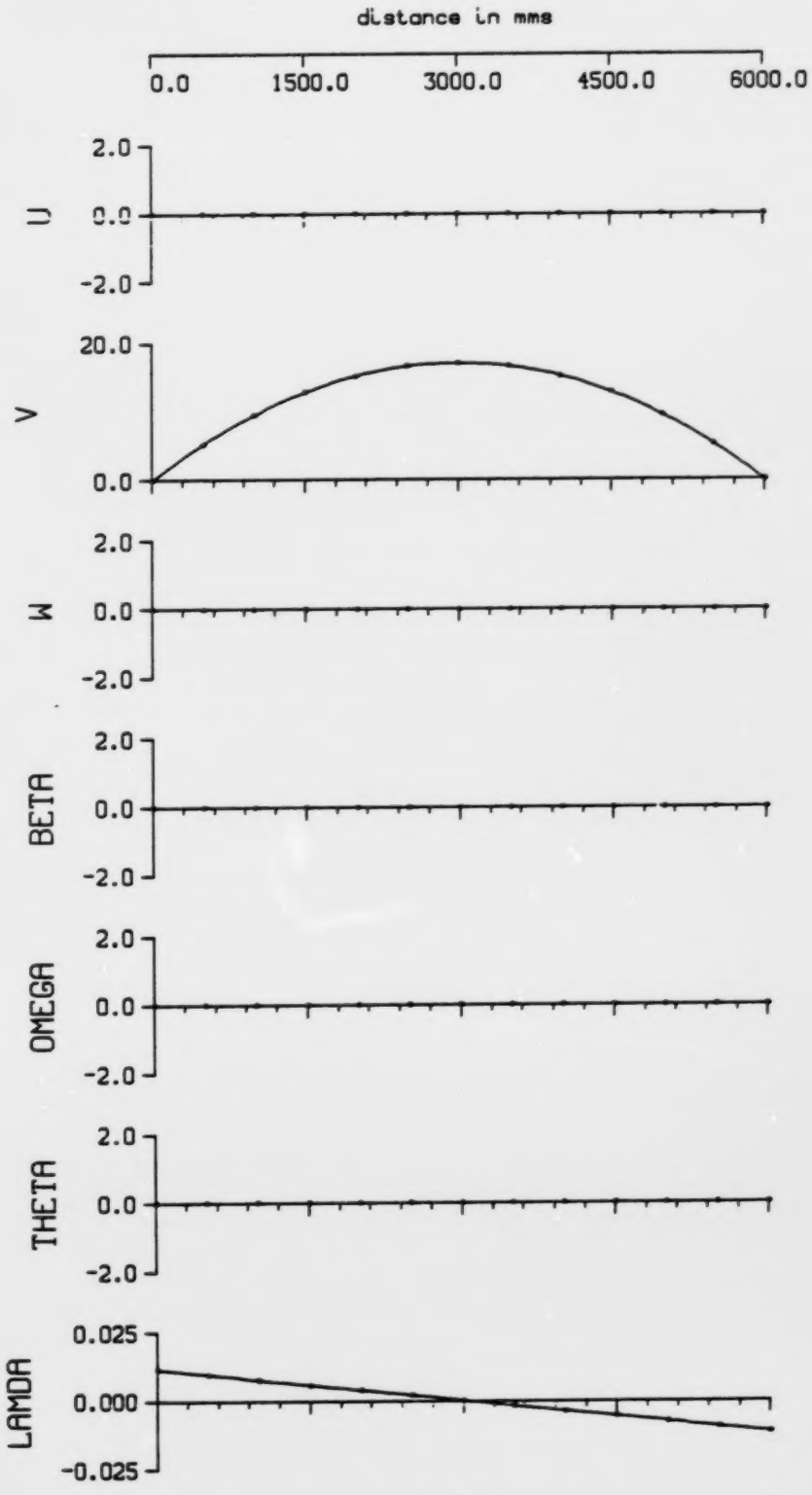

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Fig. 5

Buckling Modes Along Beam Shear Centre
(Dimensionless)

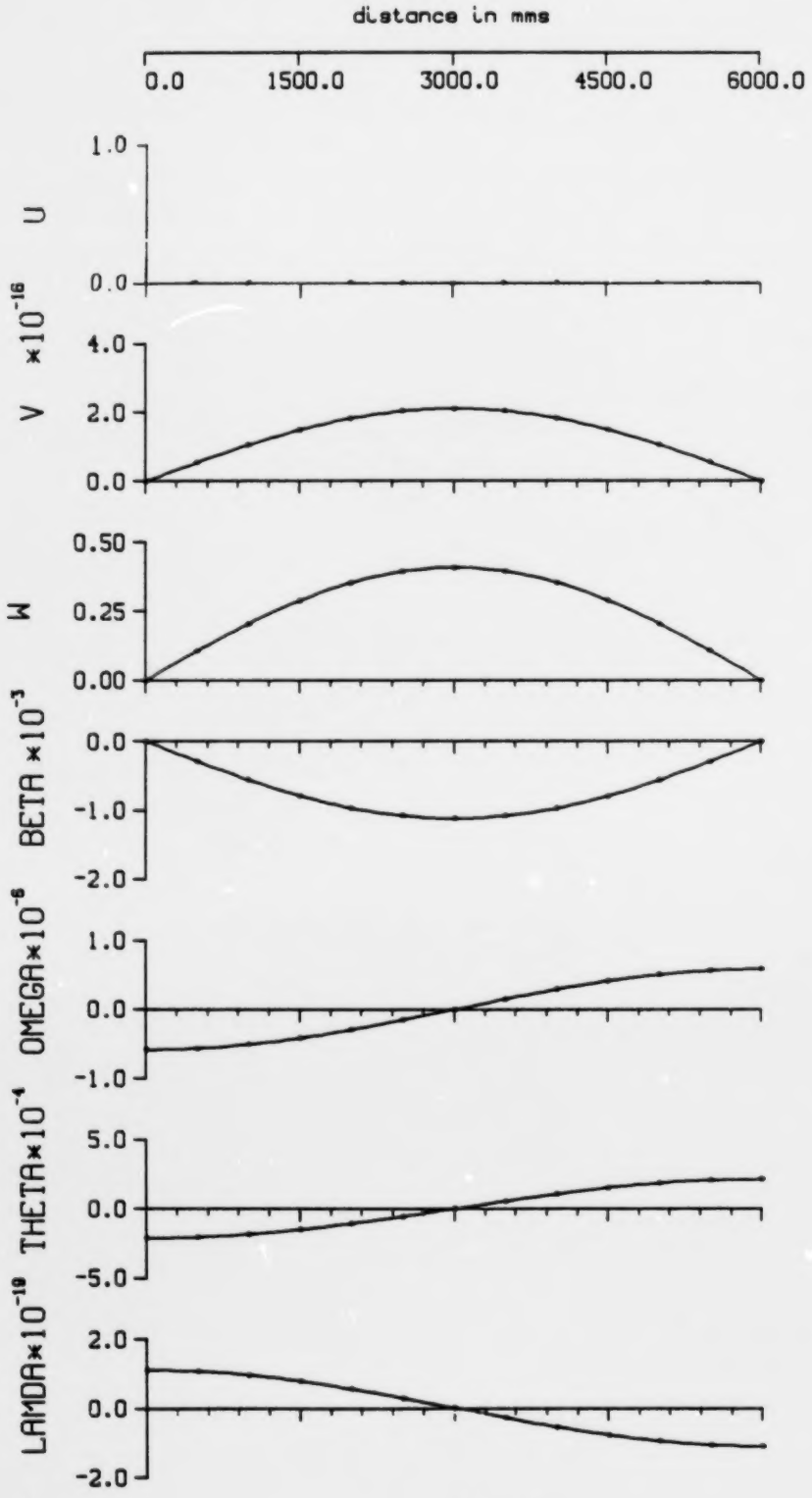

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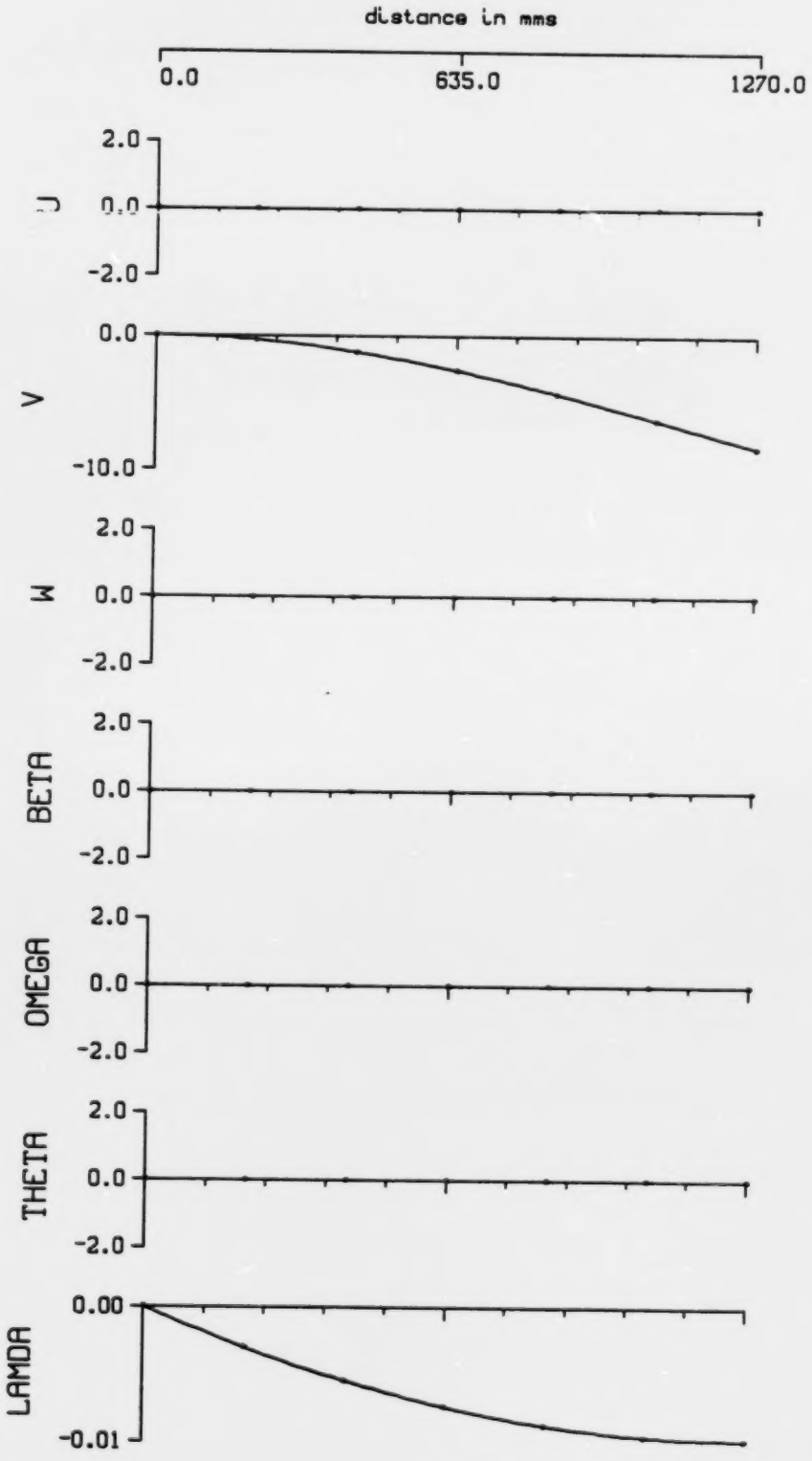
Fig. 6

Displacements Along Beam Centroid

(in mms ,radians and radians per mm)



PLOT 1 18.11.03 THUR 28 DEC, 1985 JOB-NAIL , University of Waterloo DISSPLA 9.2

Fig. 7

Buckling Modes Along Beam Shear Centre
(Dimensionless)

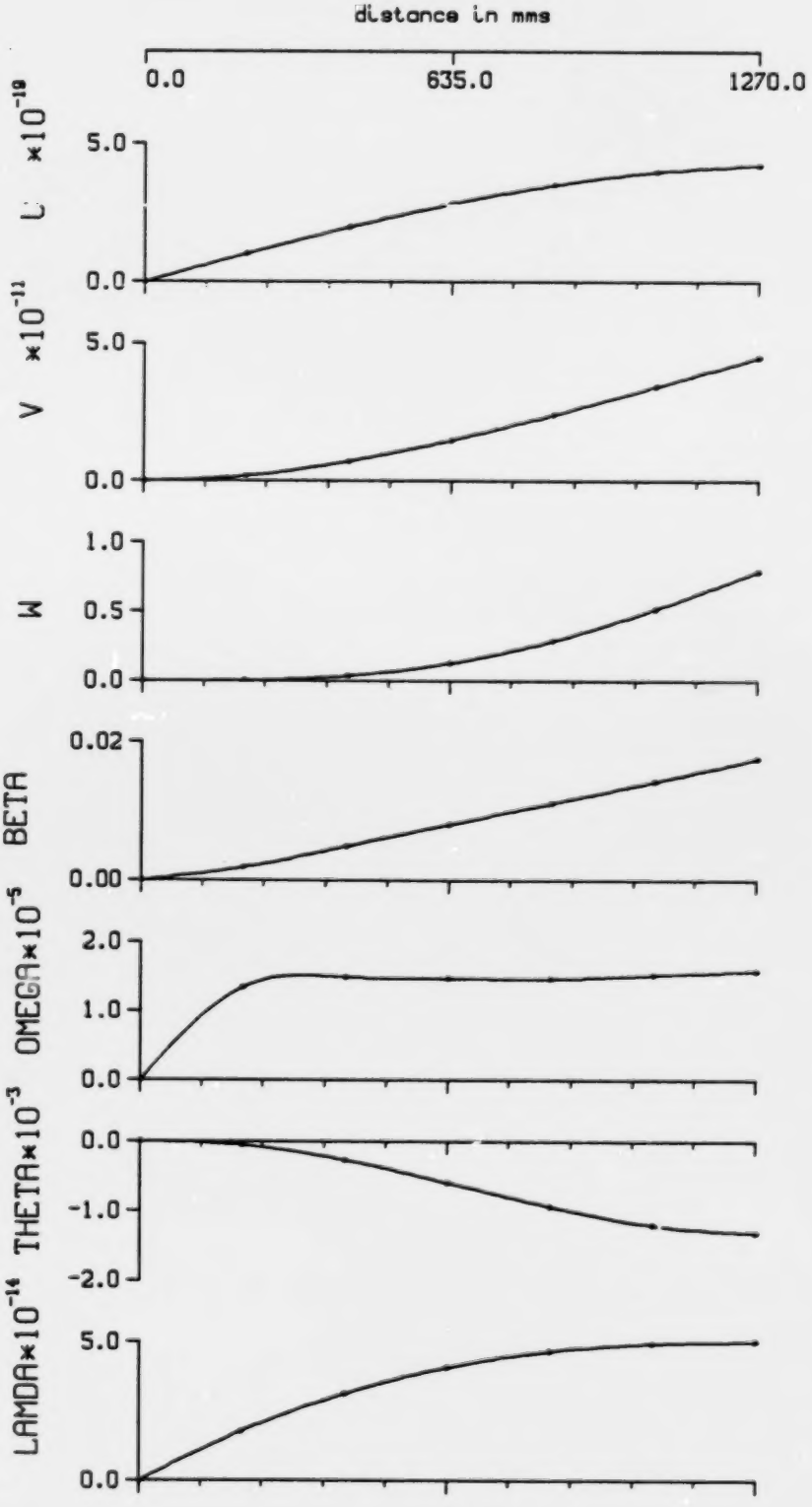


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Fig. 8

7. Stability of Rails

a) Under loads

The same example at the end of Part III is repeated here with the intention of finding the level at which the rail loses its stability under increasing central load. The reference load vector at node 16 is taken as

$$\{P^{16}\}^T_{ref} = \{20, -100, 20, 26.66, 40, -10, -49.46\}$$

The critical load level was found to be

$$\{P^{16}\}_{critical} = 1.76\ \{P^{16}\}_{ref.}$$

Figure 9 shows the pre-buckling displacement of the centroid. Figure 10 shows the distribution of the pre-buckling, internal forces along the rail centroid. Figure 11 displays the buckling modes as obtained along the rail shear centre. The buckling mode indicates an interactive lateral-torsional buckling mode.

If the vertical foundation stiffness of the middle third is reduced to 10 kip/in instead of 100 kip/in, the critical load is reduced to

$$\{P^{16}\}_{critical} = 1.56\ \{P^{16}\}_{ref.}$$

Figures 12,13 show the pre-buckling displacements and the post buckling modes for this case.

b) Under uniform temperature rise

A uniform temperature rise can be represented by two end thrusts acting through the centroid. Because a wave-type buckling mode is anticipated, the ends of the rail segment (of length 660 in.) are considered simply supported in order to eliminate the effect of finite rail length. The stability analysis predicts that buckling under uniform temperature rise occurs at N_{cr} = 1075 kips which, for E = 28900 ksi and for $\alpha = 12 \times 10^{-6}/°C$ gives

$$\Delta T_{cr} = 307\ °C$$

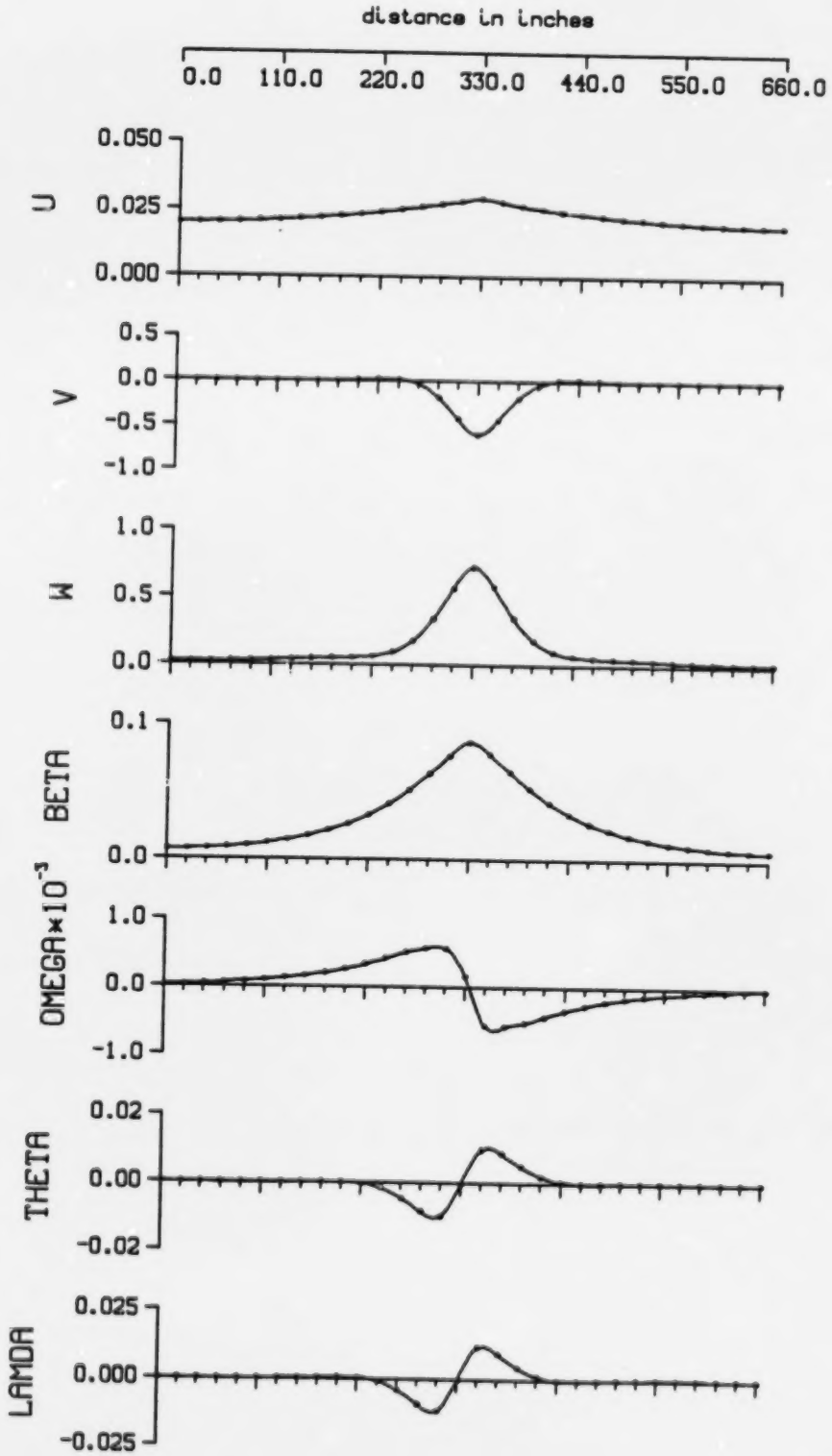
The buckling mode shown in Figure 14 suggests a lateral-torsional mode.

If the stiffness of the vertical springs over the middle third of the rail is reduced to 10 kip/in, it was found that ΔT_{cr} takes the same value as above. This is simply because the buckling mode is lateral-torsional in which the influence of the vertical springs is almost negligible. Figure 15 shows the buckling modes for this case.

In case of combined loading and temperature effect, the program first performs stability analysis for the rail assuming a uniform temperature rise. If the rail remains stable under temperature rise, the external loads are superimposed according to a pre-specified loading history and the buckling load due to temperature rise and loading is then calculated.

Displacements Along Rail Centroid

(in inches ,radians and radians per inch)



PLOT 1 13.20.15 THUR 2 JAN, 1986 JOB-ROIL , University of Waterloo DISSPLA 9.2

Fig. 9

Internal Forces Along Rail Centroid

(in Kips,Kips.inch and Kips.inch.inch)

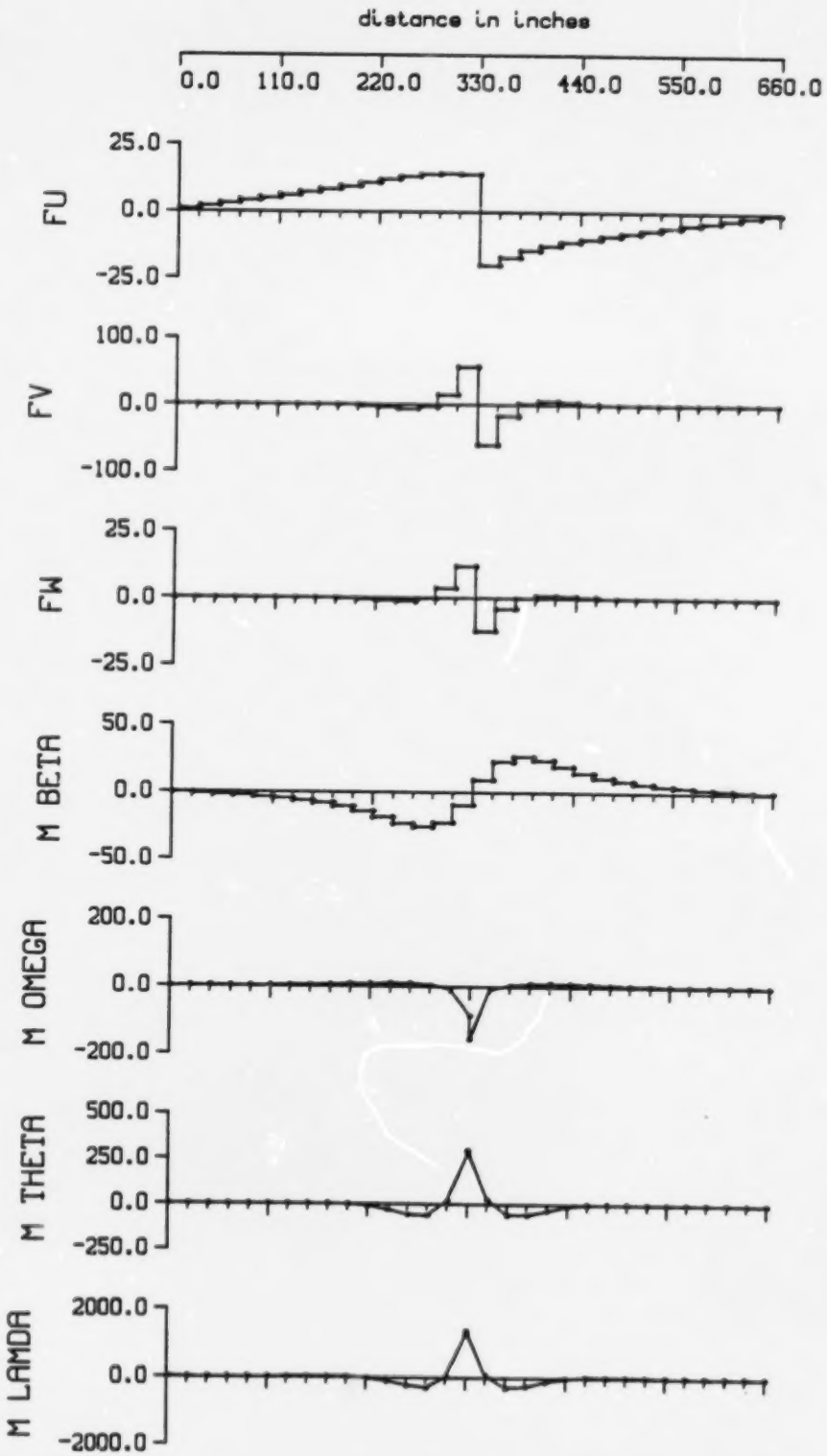


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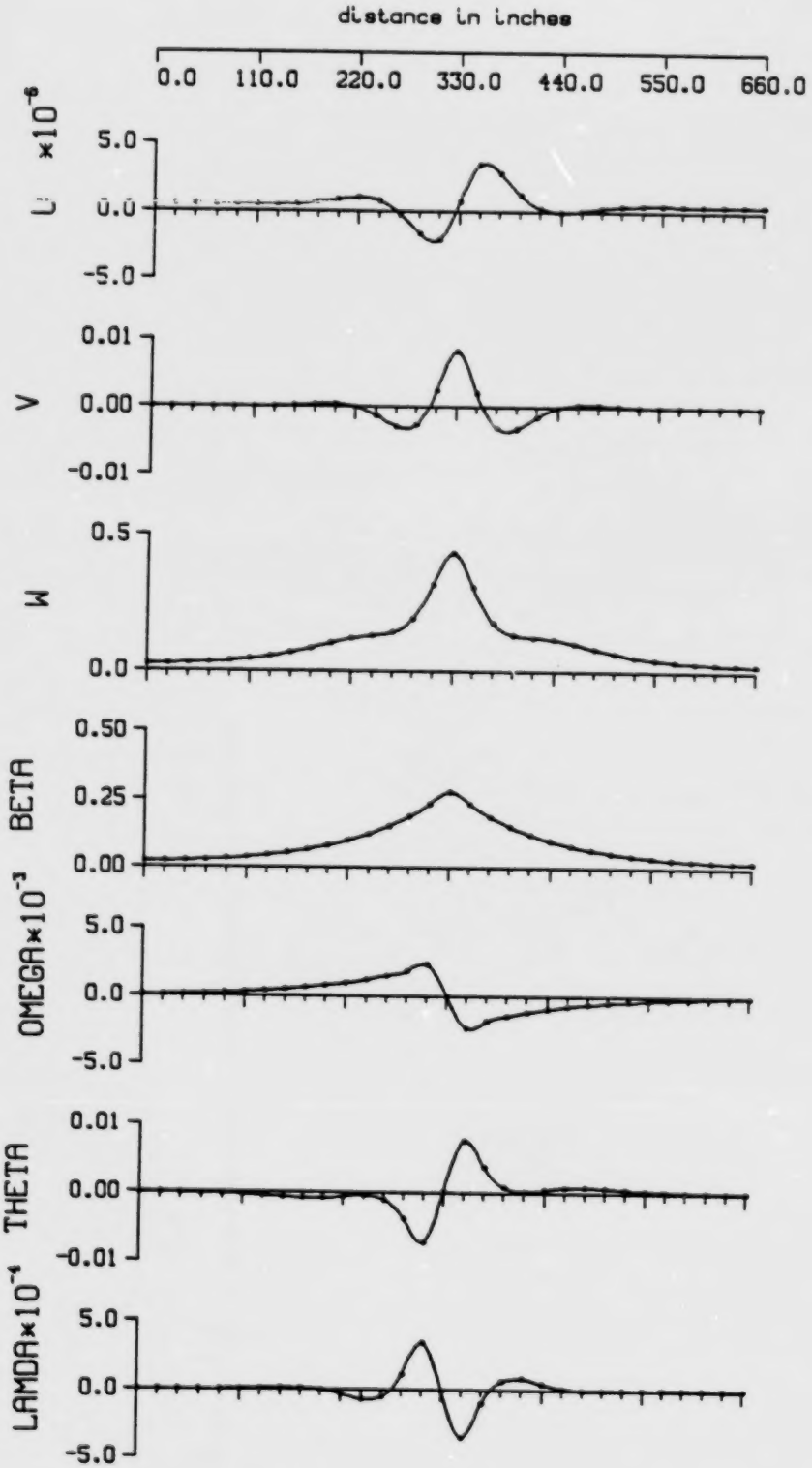
Fig. 10

Buckling Modes Along Rail Shear Centre
(Dimensionless)

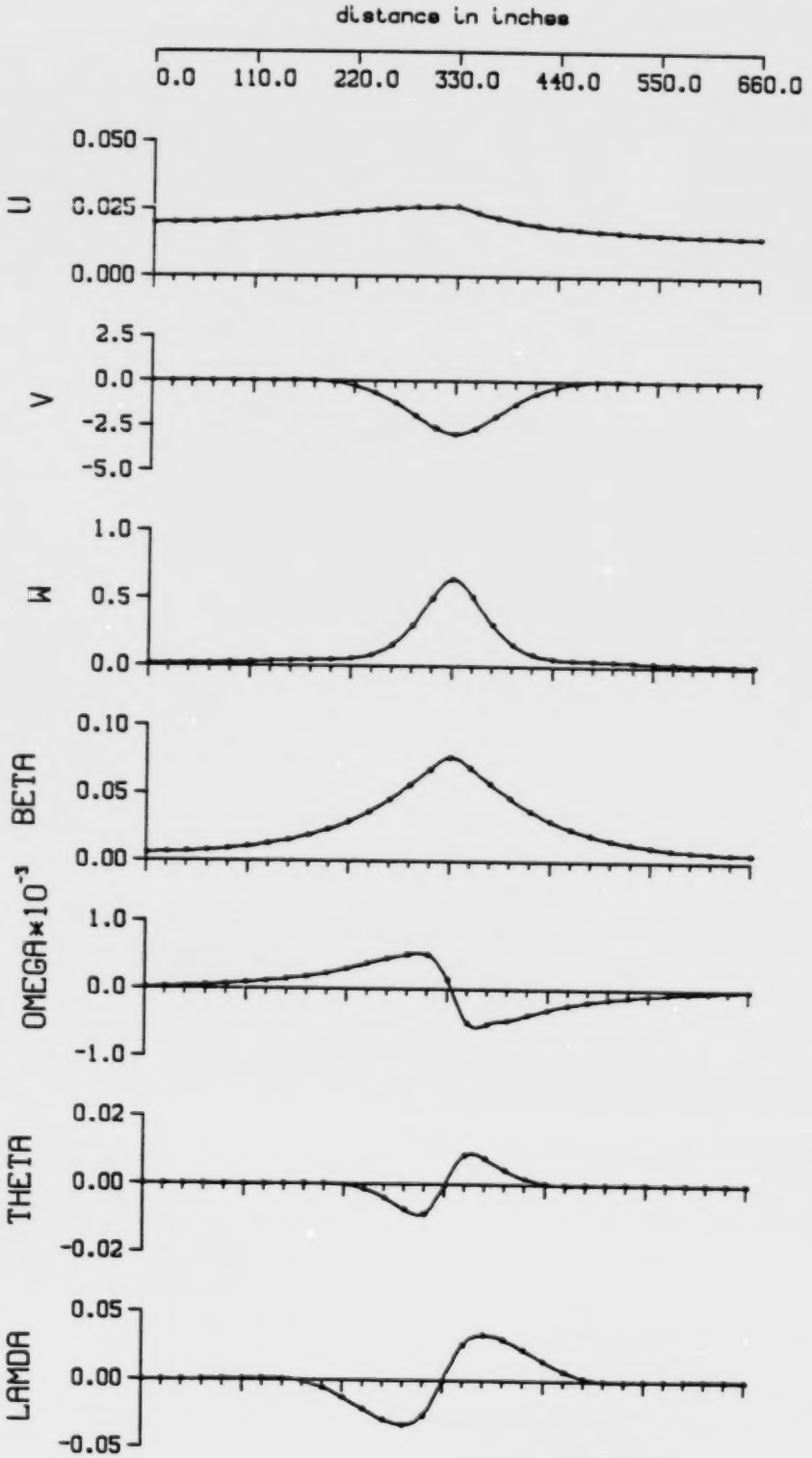


PLOT 3 13.20.53 THUR 2 JAN, 1988 JOB-RAIL , University of Waterloo DISSPLA 9.2

Fig. 11

Displacements Along Rail Centroid

(in inches ,radians and radians per inch)



PLOT 1 13.12.21 THUR 2 JAN, 1986 JOB-RAIL , University of Waterloo DISSPLA 9.2

Fig. 12

Buckling Modes Along Rail Shear Centre
(Dimensionless)

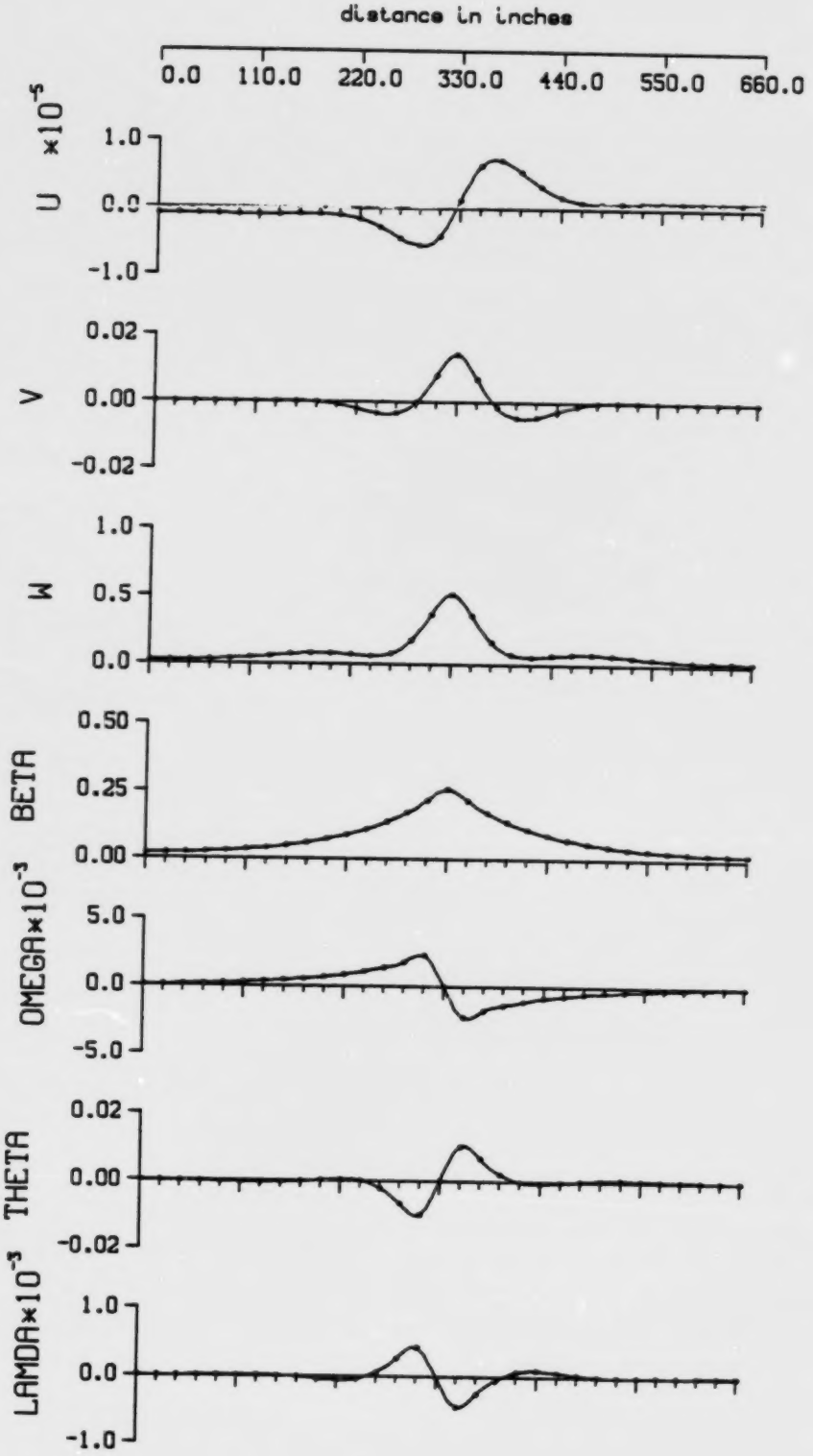


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Fig. 13

Buckling Modes Along Rail Shear Centre
(Dimensionless)

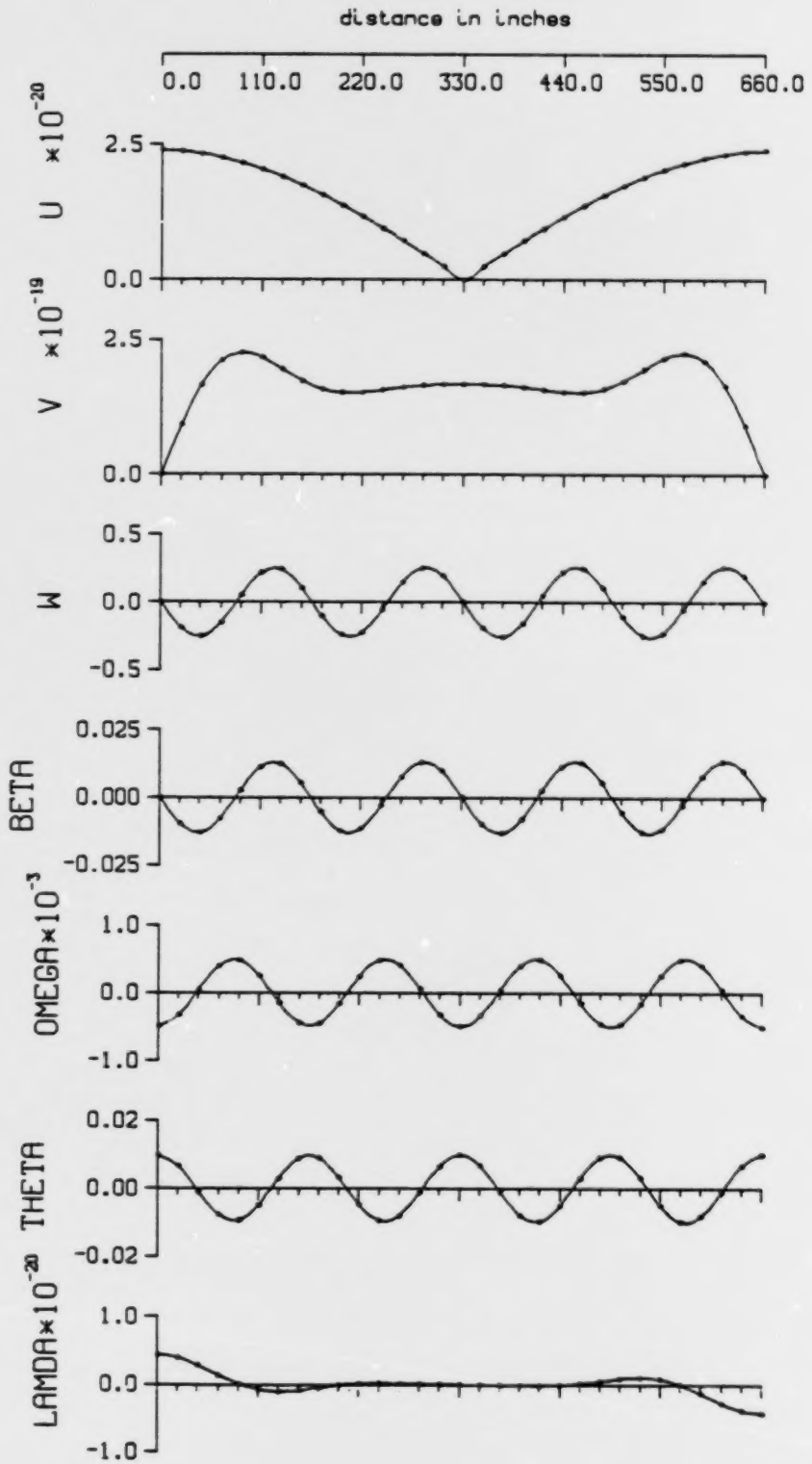

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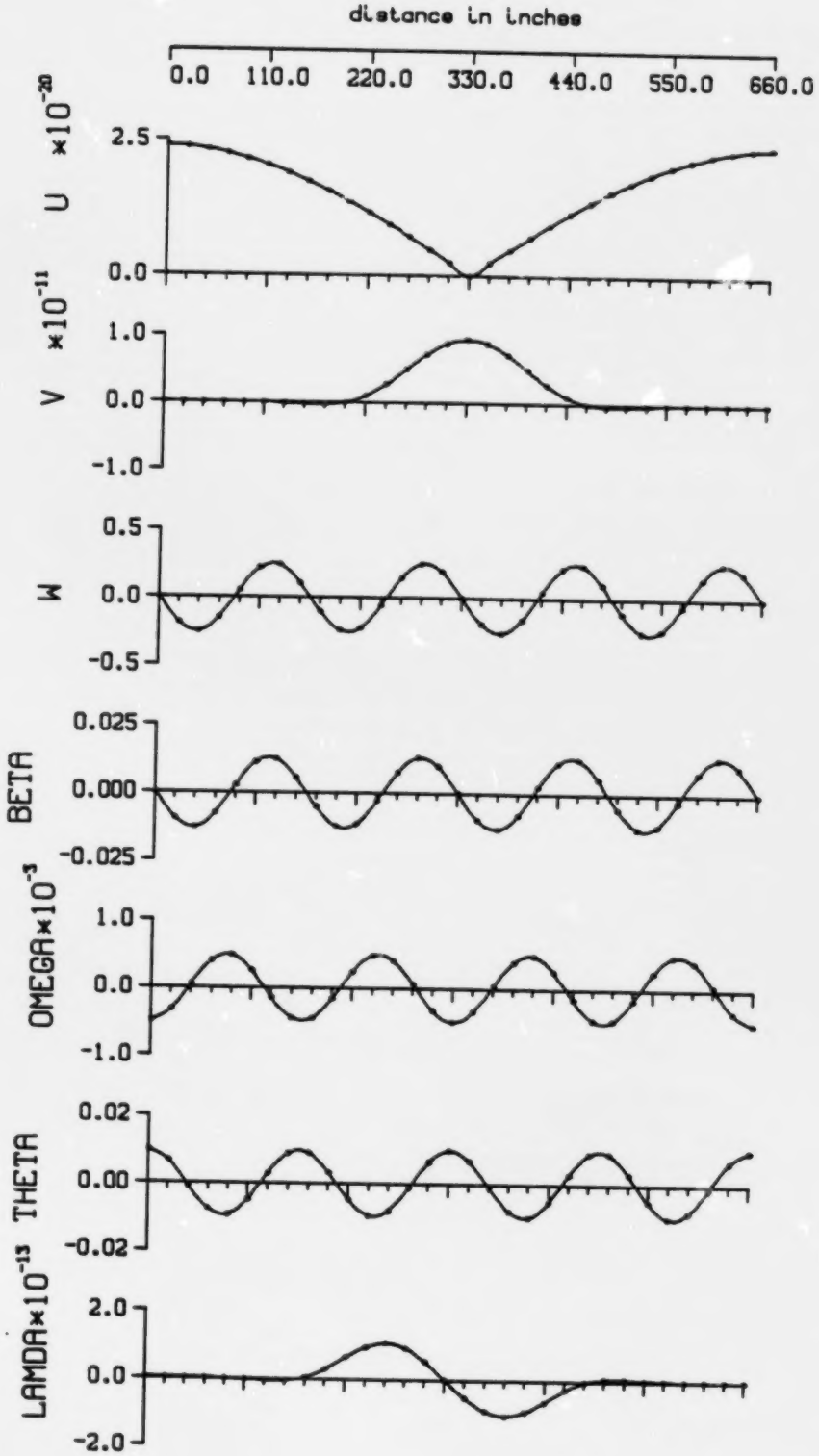
Fig. 14

Buckling Modes Along Rail Shear Centre
(Dimensionless)



PLOT 3 11.50.21 THUR 2 JAN, 1986 JOB-RAIL , University of Waterloo DISSPLA 9.2

Fig. 15

V. CONCLUSIONS

1) The Finite Element program described in Parts III and IV is designed to predict the stresses and buckling load of an infinitely long rail acted upon by up to three load components, vertical, lateral and longitudinal, acting at a discrete location. The length of the rail, for purposes of numerical analysis, is so chosen that all deformations and stresses effectively damp out over a large but finite distance. This characteristic length is also a function of the foundation reactions acting on the rail of which there are four components, one associated with each component direction, longitudinal, vertical and lateral, and one associated with torsional resistance of the anchorage between rail and tie. This last resistance is usually mobilized against potential rail overturning.

The analysis is premised upon a linear elastic theory involving elastic material response and small deformations of the rail. While the effects of all prebuckling stresses are included in the stability calculations, the variations of these stresses with the displacements is excluded. Consequently, the buckling load as calculated represents a bifurcation about an undeformed reference position.

The stress analysis calculates up to seven displacement components (3 linear and 4 rotational) about the shear centre of the rail and hence defines up to seven components of generalized forces which are used to define completely the stresses in the rail--total normal and total shear--in the prebuckling equilibrium configuration. These

generalized displacements and stresses include the warping of the rail cross-section and all additional stress components (1 normal and 2 shear) thus generated.

The F.E. program also has the added advantage of flexibility inasmuch as it allows various combinations and values of load and resistance to be easily evaluated representing variations in the physical conditions affecting subgrade, ballast and ties on the one hand and ratios of longitudinal to lateral to vertical loads for various states of track imperfection on the other. To the extent that mean values of these parameters can be defined by measurement or experience, the stress and stability analysis will yield reasonable deterministic solutions.

The analyses for strength and stability can also accommodate the effects of temperature acting separately or in conjunction with applied load. For a train of wheels whose magnitudes (mean values) and spacings are known, the stress analysis simply superimposes the individual (single load) linear solutions to determine the highest rail stresses within the loaded length. For calculation of the buckling load and mode, however, this superposition must occur before overall buckling is determined. The individual buckling solutions cannot be superimposed to yield an overall assessment of the stability problem under complex loads.

2) Although the stochastic analysis outlined in Part II makes use of data obtained from FAST, it illustrates the process to be used not only in deriving the statistical parameters of mean value and

standard deviation from limited traffic data across a control section of track, but also shows how the deformation response of the track can be calculated from these loadings and expressed in a similar statistical fashion. The analysis also shows how the F.E. Method can be programmed in a stochastic format by reference to one or two simple examples involving interactions between one or two loads and an elastic foundation also defined stochastically. This entire approach merits more detailed examination if a unified basis is sought to the total problem of the probablistic analysis of tracks.

The present investigation represents a preliminary approach to the global problem defined in the introduction of this report. Before simplifying the results or reducing them to a form suitable for the maintenance engineer, the following items might be investigated in more detail. In short, they form some suggestions for further R & D.

1) The spring constants representing the foundation reactions need to adequately model the integrated effects of tie, ballast and subgrade in the component directions. Friction and three-dimensional confinement should be included where warranted, as should the nature of the subgrade, depth and nature of ballast, geometry of the shoulders, drainage, anchorage of rail, etc. This can be approached by developing subroutines which specify not only the aggregated spring effects but also possible variations reflecting changing conditions along the track.

2) The effects of the rail deformation on the buckling load need to be investigated to account for any potential reductions from current predictions. In the analysis to date the maximum load and

reactive effects are vertical. If lateral and lateral-torsional buckling modes predominate, then the eccentricities of the applied and reactive forces may change significantly with deformation leading to substantial variations in the stability number.

3) Having properly classified the loads and reactions both as to magnitude and variation, it is possible to cast the computer program in a probabilistic format as regards predictions of stress in the rail, its stability and also its displacements.

4) Only at this point is it worth developing user manuals, program listings, etc., which enable the problem to be simplified as regards its interpretation by railroad maintenance engineers.

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Disclaimer

The Waterloo Research Institute, Drs. H. A. El-Ghazaly, F. Arbabi and A. N. Sherbourne take no responsibility for any errors, mistakes or imperfections in the programs contained in this report nor erroneous results obtained in the running of these programs.

It would be appreciated if any errors found were reported to Drs. H. A. El-Ghazaly or A. N. Sherbourne.

Appendix A

LOAD AND MONTE-CARLO SIMULATION

a. Load Data

The following table shows the ordinates of the histogram for the vertical load with a 1.2k load interval. Since the area under the histogram representing the probability density function is slightly larger than one, the ordinates of the histogram are scaled down such that the area becomes unity. The new ordinates, i.e., frequencies, as well as the cumulative frequencies, are also shown in the table.

Railroad Load Histogram

Load	f	F	Adjstd f	Adjstd F	f.P	(P-x)p	Lognorml
0						0	
1						0	
2						0	0
3						0	0.000004
4						0	0.000133
5		0.002	0	0.001834	0	0	0.001134
6	0.015	0.017	0.013761	0.015596	0.082568	1.723129	0.004679
7	0.024	0.041	0.022018	0.037614	0.154128	2.686253	0.011198
8	0.07	0.111	0.06422	0.101834	0.513761	3.422567	0.029313
9	0.078	0.189	0.071559	0.173394	0.644036	4.799833	0.050633
10	0.076	0.265	0.069724	0.243119	0.697247	5.604406	0.040594
11	0.063	0.328	0.057798	0.300917	0.635779	6.214534	0.048978
12	0.059	0.387	0.054128	0.355045	0.649541	1.457957	0.0251
13	0.044	0.431	0.040366	0.395412	0.52477	0.708635	0.038735
14	0.049	0.48	0.044954	0.440366	0.629357	0.457431	0.060128
15	0.054	0.534	0.049541	0.489908	0.743119	0.237585	0.059544
16	0.054	0.588	0.049541	0.539449	0.79266	0.070144	0.057445
17	0.044	0.632	0.040366	0.579816	0.686238	0.001425	0.054257
18	0.046	0.678	0.042201	0.622018	0.7596..	0.027694	0.050396
19	0.043	0.721	0.039449	0.661467	0.749541	0.129253	0.046181
20	0.038	0.759	0.034862	0.69633	0.697247	0.275294	0.041779
21	0.038	0.797	0.034862	0.731192	0.73211	0.50609	0.037464
22	0.035	0.832	0.03211	0.763302	0.706422	0.74296	0.033331
23	0.03	0.862	0.027522	0.790825	0.633027	0.929096	0.02946
24	0.024	0.886	0.022018	0.812844	0.52844	1.021152	0.025898
25	0.023	0.909	0.0211	0.833944	0.527522	1.287103	0.022664
26	0.022	0.931	0.020183	0.854128	0.52477	1.566396	0.019759
27	0.022	0.953	0.020183	0.874311	0.5[illegible]954	1.942416	0.017174
28	0.02	0.973	0.018348	0.89266	0.513761	2.144185	0.014889
29	0.018	0.991	0.016513	0.909174	0.478899	2.30331	0.012881
30	0.015	1.006	0.013761	0.922935	0.412844	2.258235	0.011125
31	0.014	1.02	0.012844	0.935779	0.398165	2.449597	0.009596
32	0.013	1.033	0.011926	0.947706	0.381651	2.615967	0.008268
33	0.012	1.045	0.011009	0.958715	0.363302	2.751842	0.007119
34	0.011	1.056	0.010091	0.968807	0.343119	2.851716	0.006125
35	0.01	1.066	0.009174	0.977981	0.3211	2.910085	0.005268
36	0.009	1.075	0.008256	0.986238	0.297247	2.921445	0.00453
37	0.007	1.082	0.006422	0.99266	0.237614	2.520255	0.003895
38	0.003	1.085	0.002752	0.995412	0.104587	1.191908	0.003349
39	0.003	1.088	0.002752	0.998165	0.107339	1.309211	0.00288
40	0.002	1.09	0.001834	1	0.073394	0.954679	0.002478
20					17.1899	60.59514	
						7.784288	

Lognormal a,b 0.431894 2.751055
Gumbel a,b 0.164761 13.68787

Gumbel	Normal
0.000113	0.004474
0.000409	0.005893
0.001185	0.007635
0.00285	0.00973
0.00585	0.012196
0.010498	0.015038
0.016813	0.018238
0.024447	0.021756
0.032752	0.025529
0.040934	0.029466
0.048233	0.033453
0.054065	0.037358
0.058089	0.041036
0.060209	0.044338
0.060533	0.047122
0.059306	0.049261
0.056845	0.050654
0.053482	0.051234
0.049531	0.050972
0.045263	0.049882
0.040896	0.048016
0.036597	0.045464
0.032484	0.042342
0.028634	0.03879
0.025093	0.034953
0.02188	0.030981
0.018999	0.027011
0.016438	0.023164
0.014179	0.019539
0.012199	0.016212
0.010473	0.013232
0.008974	0.010622
0.007678	0.008388
0.00656	0.006515
0.005599	0.004977
0.004774	0.00374
0.004067	0.002765
0.003462	0.00201
0.002946	0.001438
0.002505	0.001011
0.00213	0.0007

b. Representation of data

Using the moments of the values given in the above table, the parameters for the different distributions are obtained. In order to choose the most appropriate distribution and have a visual comparison, a computer program, PLOT, was written that plots the histogram and calculates and superimposes a variety of distributions on this plot. The output of this program is shown in Figures 1 to 3. A listing of this program for an IBM-PC is given below.

Note that these values are somewhat different from the mean and standard deviation used in the analysis. However, only one set of data was considered for the analysis and due to a large number of sample points involved, some of the values were taken from the histogram. This of course is not as accurate as using the original data. Furthermore, since the presentation here is intended to demonstrate the suggested procedure and how it should be implemented in a finite element program, it does not seem necessary to modify the numerical examples to reflect the present values. The same values of mean and standard deviation were consistently used in all the numerical examples. For a parametric study more data must be considered with a more comprehensive model rather than the simple solutions used here.

```
10 PRINT "x-y plot" :KEY 10,"RUN 190"+CHR$(13)
20 INPUT "plot input from keyboard?(y/cr)",I$
30 IF I$<>"y" THEN GOTO 110
40 SCREEN 1,1
50 INPUT "graph lable: ",GL$
60 INPUT "x axis lable: ",XL$
70 INPUT "y axis lable: ",YL$
80 INPUT "Min x , Max x , x div: ",X1,X2,NX
90 INPUT "Min y , Max y , y div; ",Y1,Y2,NY
100 GOTO 260
110 CLS
120 LIST 130-230
130 'input the following then press f10
140 'GRAPH LABLE
150 'X AXIS LABLE
160 'Y AXIS LABLE
170 'X MIN,MAX,DIVISIONS
180 'Y MIN,MAX,DIVISIONS
190 DATA Load Distribution
200 DATA Axle Load
210 DATA Distribution
220 DATA 0,40,5
230 DATA 0,.1,5
240 SCREEN 1
250 READ GL$,XL$,YL$,X1,X2,NX,Y1,Y2,NY
260 CLS
270 AX1=60 :AX2=300 :AY1=160 :AY2=40
280 LX=AX2-AX1 :LY=AY1-AY2
290 LINE (AX1,AY1)-(AX2,AY2),,B
300 N2=CINT(AX1/8)+LX/16-8 :N1=AY2/8-3
310 LOCATE N1,N2
320 PRINT GL$
330 N1=AX1/8+LX/16 :N2=AY1/8-2
340 LOCATE N1,N2
350 PRINT XL$
360 NC=LEN(YL$)
370 FOR I=1 TO NC
380 N2=CINT(AX1/8)-5 :N1=AY2/8+I
390 LOCATE N1,N2
400 CH$=MID$(YL$,I,1)
410 PRINT CH$
420 NEXT
430 DX1=LX/NX
440 DY1=LY/NY
450 N=NX :A1=AX1 :B1=AY1
460 DX=DX1 :DY=0 :D1=0 :D2=-2
470 GOSUB 980
480 B1=AY2 :D2=2
490 GOSUB 980
500 N=NY :B1=AY1
510 DY=-DY1 :DX=0 :D1=2 :D2=0
520 GOSUB 980
530 A1=AX2 :D1=-2
540 GOSUB 980
550 DDX=(X2-X1)/NX :DDY=(Y2-Y1)/NY
560 N1=AY1/8+2
570 FOR I=1 TO NX+1
580 II=(I-1)*DX1/8+AX1/8-2
590 I1=CINT(II)
600 LOCATE N1,II
610 PRINT X1+(I-1)*DDX;
620 NEXT
630 N1=AX1/8-4
```

```
640 FOR I=1 TO NY+1
650 II=AY1/8-(I-1)*DY1/8+1
660 LOCATE II,N1
670 PRINT USING ".##"; Y1+(I-1)*DDY
680 NEXT
690 DATA 5,.002,6,.014,7,.022,8,.064,9,.072,10,.07,11,.058,12,.054,13,.04,14,.04
5
700 DATA 15,.05,16,.05,17,.04,18,.042,19,.039,20,.035,21,.034,22,.032,23,.028
710 DATA 24,.022,25,.021,26,.02,27,.02,28,.018,29,.017,30,.014,31,.013,32,.012
720 DATA 33,.011,34,.01,35,.009,36,.008,37,.006,38,.003,39,.003,40,.002
730 COLOR 1,0
740 PSET(AX1+5*DX1/DDX,AY1)
750 FOR I=1 TO 35
760 READ X,Y
770 X=AX1+X*DX1/DDX :Y=AY1-Y*DY1/DDY
780 LINE -(X,Y)
790 LINE -(X+DX1/DDX,Y)
800 NEXT
810 N=2
820 FOR J=1 TO N
830 IF J=1 THEN A=.4325519:IF J= 1 THEN B=2.748448
840 IF J=2 THEN A=.164922 :IF J=2 THEN B=13.65138
850 IF J=3 THEN A=.4325519:IF J= 3 THEN B=2.748448 :IF J=3 THEN C=15.10169
860 DEF FND(X)=1/SQR(2)/SQR(3.14159)/A/X*EXP(-.5*((LOG(X)-B)/A)^2)
870 DEF FNG(X)=A*EXP(-A*(X-B)-EXP(-A*(X-B)))
880 COLOR 1,2
890 FOR X=1 TO 40 STEP .25
900 XX=AX1+X*DX1/DDX
910 IF J=1 THEN F=FND(X)
920 IF J=2 THEN F=FNG(X)
930 Y=AY1-F*DY1/DDY
940 PSET(XX,Y)
950 NEXT
960 NEXT J
970 END
980 FOR I=1 TO N
990 A=I*DX+A1 :B=I*DY+B1
1000 PSET(A,B)
1010 X=A+D1 :Y=B+D2
1020 LINE (A,B)-(X,Y)
1030 NEXT
1040 RETURN
```

c. Monte-Carlo Simulation

When data is not available for a random variable, then a Monte-Carlo analysis can be performed. The process begins by generating a set of random numbers with a uniform distribution. In many cases, even in the absence of random data, their distribution can be known or assumed based on the physics of the phenomenon or the characteristics of the process. The set of random numbers with a uniform distribution is then transformed into another set with a specified distribution. The latter is subsequently used to perform the analysis of the structure. Once response of the structure to this random set is found, the sample statistics and histogram of the response can be calculated.

Since the sample size must be large enough to make the results meaningful, this is best done by means of a computer program. Such a program was written with the provision that any distribution can be specified in a statement function. The steps involved in this procedure are as follows:

1. Input the number of random values to be generated and the number of divisions for the numerical integration of the distribution.
2. Generate a set of random values between zero and one with a uniform distribution.
3. Calculate the ordinates of the specified distribution density function.
4. Perform numerical integration of the specified distribution density function in order to obtain the ordinates of the cumulative distribution function.
5. Using the generated random numbers with a uniform distribution

as the ordinates of the specified cumulative distribution function, calculate the abscissa or the random numbers that have the specified distribution.

For a large sample size, the set of random values thus obtained may be placed on file. This file can then be used by the analysis program for calculating the response. The statistics of the response and its histogram can then be easily obtained.

Since the above procedure can be used for a Monte-Carlo simulation of any problem, the above procedure was programmed in a general form for an IBM-PC using the Basic language and for a Hewlett-Packard HP41cv programmable calculator. Because of the similarity of Basic and Fortran languages, only a few statements in the program need to be slightly modified in order to obtain a Fortran program. The listing of the Basic program called RSTOCH is given below. A compiled version of this program called MONTCRLO is also available and should preferably be used because, for large sample sizes, it can run much faster than the interpretive version.

The results of three different runs using MONTCRLO program are shown on the following pages. The first result is for a sample size of 50 with 20 divisions of the distribution curve. The last two are with 50 divisions of the distribution curve and with sample sizes 100 and 200. These results are compared to the measured values in the following table.

Sample Size	50	100	200	From measurements
Mean Value	20.67	19.33	18.84	18.4
Standard Dev.	5.56	5.48	5.78	8.49

```
10 PRINT "Random Numbers for Specified Distribution" :PRINT :PRINT
20 INPUT "Do you want a hard copy? (y/n)",H$:P=0 :PRINT
30 IF H$ = "y" OR H$="Y" THEN 50
40 GOTO 60
50 P=1
55 IF P<>0 THEN LPRINT "Random Values for Specified Distribution"
56 IF P<>0 THEN LPRINT "----------------------------------------":LPRINT :LPRINT

60 REM *******Distribution density functions is Lognormal as given below******
70 PI= 3.14159 : A=.310994 : B=2.881341
80 DEF FND(X)=1/SQR(2*PI)/A/X*EXP(-.5*((LOG(X)-B)/A)^2)
90 INPUT "No. of Div. to describe the distrib. (Max 100,Default 20)";ND:PRINT
100 IF P<>0 THEN LPRINT "No of Div. of the Distribution func.(Default 20)";ND:
PRINT
110 INPUT "Do you want a printout of the frequency distibution function and
          cummul. distribution function? (y/n)",PD$:PRINT:IF ND=0 THEN ND=20
120 INPUT "Max value of the random var.(Default 40)";S:PRINT:IF S=0 THEN S=40
130 IF P<>0 THEN LPRINT "Max value of the random variable";S :LPRINT
140 INPUT "No.of rand. val. gen.(Max 1000 Def. 50";NR:PRINT:IF NR=0 THEN NR =50
150 IF NR=0 THEN NR=50
160 IF P<>0 THEN LPRINT "Sample Size";NR :LPRINT
170 INPUT "Do you want a list of the random numbers generated? (y/n)";PR$:PRINT
180 DIM DOR(101),F(101),R(1001),XI(1001)
190 PRINT "Generate ordinate of the probability density fynction":PRINT
200 DX=S/ND :X=0
210 FOR I=1 TO ND+1 :IF X<.000001 THEN X=.000001
220 DOR(I)=FND(X)
230 X=X+DX : NEXT I
240 PRINT "Integrate to get the ordinate of the distribution function":PRINT
250 F(1)=0 :X=DX
260 PRINT "Rand. Value ";"   Ord. of freq. dis. func.","Ord. of cum. dis. fun

270 IF P<>0 THEN LPRINT "Rand. Value";"    Ord.  freq. dis. func.","Ord.  cum.
is. func."
280 IF PD$="y" OR PD$="Y" THEN PRINT 0,DOR(1),F(1)
290 IF P<>0 AND (PD$="y" OR PD$="Y") THEN LPRINT 0,DOR(1),F(1)
300 FOR I=2 TO ND+1 :SUM=(DOR(I)+DOR(I-1))/2*DX :F(I)=F(I-1)+SUM
310 IF PD$ ="y" OR PD$="Y" THEN PRINT X,DOR(I),F(I)
320 IF P<>0 AND (PD$ ="y" OR PD$="Y") THEN LPRINT X,DOR(I),F(I)
330 X=X+DX :NEXT I
340 PRINT "The last ordinate of the cumm. density func. is set =1"
350 IF P<>0 THEN LPRINT "The last ordinate of the cumm. dens. func. is set =1
LPRINT :LPRINT
360 F(ND+1)= 1
370 FOR I=1 TO NR :R(I)=RND :NEXT I
380 GOSUB 570
390 BEEP :PRINT "Interpolate to get random var. x(i) w/ given distr." :PRINT
400 PRINT "Sorted random values   Distributed random values" :PRINT
410 IF P<>0 THEN LPRINT "Sorted random values   Distributed random values"
420 J=1
430 FOR I=1 TO NR
440 IF R(I)>F(J) AND R(I)<F(J+1) THEN 460
450 J=J+1 :IF J<=ND THEN 440
460 XI(I)=DX*J+(R(I)-F(J))/(F(J+1)-F(J))*DX
470 IF PR$="y" OR PR$="Y" THEN PRINT R(I),XI(I)
480 IF P<>0 AND (PR$="y" OR PR$="Y") THEN LPRINT R(I),XI(I)
490 J=J-1  :NEXT I
500 PRINT "Mean and std.dev. of the Distributed ramdom values"
510 SUM=0 :SUM1=0
520 FOR I=1 TO NR :SUM=SUM+XI(I):SUM1=SUM1+XI(I)^2 :NEXT I
530 MEAN=SUM/NR :VAR=SUM1/NR-MEAN^2 :SD=SQR(VAR)
540 PRINT "mean =";MEAN,"std dev.=";SD
550 IF P<>0 THEN LPRINT "mean =";MEAN,"std dev.=";SD
```

```
560 END
570 PRINT "Sorting the random values"
580 SS=0
590 FOR I=1 TO NR-1
600 IF R(I)<=R(I+1) THEN 620
610 SS=SS+1 :SWAP R(I),R(I+1)
620 NEXT I
630 IF SS>0 THEN 580
640 RETURN
```

Random Values for Specified Distribution

No of Div. of the Distribution func.(Default 20) 0

Max value of the random variable 40

Sample Size 50

Rand. Value	Ord. freq. dis. func.	Ord. cum. dis. func.
0	0	0
2	1.139729E-11	1.139729E-11
4	3.074413E-06	3.074435E-06
6	4.619148E-04	4.680636E-04
8	5.771975E-03	6.701953E-03
10	2.270481E-02	3.517874E-02
12	4.743782E-02	.1053214
14	6.764495E-02	.2204041
16	7.541964E-02	.3634687
18	.0712365	.5101249
20	5.994449E-02	.6413059
22	4.645202E-02	.7477024
24	3.390666E-02	.828061
26	2.368613E-02	.8856539
28	1.601789E-02	.9253579
30	1.057492E-02	.9519507
32	6.858698E-03	.9693843
34	4.391103E-03	.9806341
36	2.785248E-03	.9878104
38	1.755266E-03	.9923509
40	1.101483E-03	.9952077

The last ordinate of the cumm. dens. func. is set =1

Sorted random values	Distributed random values
.0237478	11.19718
2.613592E-02	11.3649
5.589956E-02	12.59082
7.355577E-02	13.09426
.1064997	14.02048
.1565262	14.88988
.1677659	15.08521
.17808	15.26446
.1881671	15.43976
.2190234	15.976
.2868863	16.9294
.3085267	17.23193
.3366159	17.62461
.3747999	18.15453
.3797226	18.22166
.3887324	18.34453
.3899929	18.36172
.393975	18.41603
.4001461	18.50018
.4310312	18.92137
.4338114	18.95929
.4422875	19.07488
.4441276	19.09997
.4694337	19.44508
.4963169	19.8117
.4982019	19.8374

.5062703	19.94743
.5271801	20.26003
.5274349	20.26391
.5309483	20.31748
.543805	20.51349
.5770086	21.01972
.6045134	21.43906
.6321052	21.85973
.6424253	22.02104
.6529394	22.21868
.6757547	22.64756
.7473214	23.99284
.7886396	25.01886
.7974609	25.23841
.8178547	25.74598
.8267782	25.96807
.8275138	25.98638
.8280919	26.00107
.8317046	26.12653
.8798236	27.79754
.8854808	27.99399
.9512645	31.94839
.9602815	32.95572
.9944627	40.55217

mean = 20.67383 std dev.= 5.565125

Random Values for Specified Distribution

No of Div. of the Distribution func.(Default 20) 50

Max value of the random variable 40

Sample Size 100

Rand. Value	Ord. freq. dis. func.		Ord. cum. dis. func.
0	0	0	
.8	3.685332E-22		1.474133E-22
1.6	7.069118E-14		2.827647E-14
2.4	4.94806E-10	1.97979E-10	
3.2	9.437434E-08		3.814564E-08
4	3.074413E-06		1.30566E-06
4.8	3.613476E-05		1.698933E-05
5.6	2.219652E-04		1.202293E-04
6.400001	8.768875E-04		5.597704E-04
7.200001	2.527997E-03		1.921724E-03
8.000001	5.771984E-03		5.241716E-03
8.800001	1.103395E-02		1.196409E-02
9.600001	.0183661	2.372411E-02	
10.4	2.738789E-02		.0420257
11.2	3.737535E-02		.067931
12	4.743786E-02		.1018563
12.8	5.670633E-02		.1435139
13.6	6.447648E-02		.1919871
14.4	7.028576E-02		.245892
15.2	.0739274	.3035772	
16	7.541962E-02		.363316
16.8	7.495083E-02		.4234642
17.6	7.281818E-02		.4825718
18.4	6.937155E-02		.5394477
19.2	6.496884E-02		.5931839
20	5.994449E-02		.6431492
20.8	5.459041E-02		.6889632
21.6	4.914688E-02		.7304581
22.4	4.380095E-02		.7676372
23.2	3.868982E-02		.8006335
23.99999	3.390667E-02		.8296721
24.79999	2.950826E-02		.8550381
25.59999	2.552217E-02		.8770502
26.39999	2.195391E-02		.8960407
27.19999	1.879296E-02		.9123395
27.99999	1.601794E-02		.9262638
28.79999	1.360057E-02		.9381112
29.59999	1.150896E-02		.948155
30.39999	9.709841E-03		.9566425
31.19999	8.170269E-03		.9637945
31.99999	6.858709E-03		.9698061
32.79999	5.745843E-03		.974848
33.59999	4.804819E-03		.9790682
34.39999	4.011558E-03		.9825948
35.19999	3.344637E-03		.9855373
35.99999	2.785253E-03		.9879893
36.79998	2.317044E-03		.9900302
37.59998	1.925857E-03		.9917274
38.39998	1.599533E-03		.9931375
39.19998	1.327686E-03		.9943084

39.99998 1.10149E-03 .99528
The last ordinate of the cumm. dens. func. is set =1

Sorted random values	Distributed random values
.0237478	10.40104
2.400047E-02	10.41208
2.613592E-02	10.50543
3.657872E-02	10.9619
5.589956E-02	11.62845
5.595702E-02	11.63022
5.670512E-02	11.65333
6.105608E-02	11.78769
7.355577E-02	12.13264
.1064997	12.88917
.1092455	12.9419
.133105	13.4001
.1421084	13.57301
.144884	13.62261
.1485015	13.68231
.1565262	13.81475
.1613985	13.89517
.1677659	14.00025
.17808	14.17048
.1881671	14.33696
.2190234	14.80125
.2336911	15.01893
.2529784	15.29828
.2652186	15.46803
.2669823	15.49249
.2868863	15.76852
.300068	15.95133
.3085267	16.06628
.3227947	16.25735
.3366159	16.44244
.3395019	16.48109
.3587979	16.73949
.3609428	16.76822
.3747999	16.95274
.3797226	17.01822
.3887324	17.13805
.3899929	17.15482
.393975	17.20778
.4001461	17.28986
.4082345	17.39744
.4310312	17.70242
.4338114	17.74005
.4422875	17.85477
.4441276	17.87967
.4604465	18.10054
.4670625	18.19009
.4694337	18.22218
.4748311	18.29523
.4963169	18.59333
.4982019	18.61985
.5062703	18.73334
.5078771	18.75594
.5156872	18.86579
.5243073	18.98704
.5271801	19.02745
.5274349	19.03103
.5309483	19.08045
.543805	19.26487
.5450828	19.28389
.5513691	19.37748
.5613249	19.5257

.5694048	19.64599
.5770086	19.75919
.6018766	20.13918
.6045134	20.1814
.6321052	20.62317
.6335627	20.64651
.6424253	20.78841
.6486098	20.89535
.6529394	20.97096
.6757547	21.36936
.6763578	21.37989
.6844444	21.5211
.6913298	21.64563
.7362378	22.52436
.7473214	22.76286
.7546886	22.92138
.7669859	23.18599
.7886396	23.70921
.7905869	23.75642
.7974609	23.92308
.8122079	24.31887
.8178547	24.47444
.8267782	24.72028
.8275138	24.74054
.8280919	24.75647
.8317046	24.8641
.8704043	26.15846
.8745503	26.30914
.8798236	26.51683
.8854808	26.75515
.8981191	27.30202
.8982948	27.31064
.9278255	28.90545
.9286163	28.95885
.9400988	29.75831
.9512645	30.69309
.9602815	31.60704
.9758157	33.78344
.9944627	40.0217

mean = 19.33588 std dev.= 5.483387

Random Values for Specified Distribution

No of Div. of the Distribution func.(Default 20) 50

Max value of the random variable 40

Sample Size 200

Rand. Value	Ord. freq. dis. func.		Ord. cum. dis. func.
0	0	0	
.8	3.685332E-22		1.474133E-22
1.6	7.069118E-14		2.827647E-14
2.4	4.94806E-10	1.97979E-10	
3.2	9.437434E-08		3.814564E-08
4	3.074413E-06		1.30566E-06
4.8	3.613476E-05		1.698933E-05
5.6	2.219652E-04		1.202293E-04
6.400001	8.768875E-04		5.597704E-04
7.200001	2.527997E-03		1.921724E-03
8.000001	5.771984E-03		5.241716E-03
8.800001	1.103395E-02		1.196409E-02
9.600001	.0183661	2.372411E-02	
10.4	2.738789E-02		.0420257
11.2	3.737535E-02		.067931
12	4.743786E-02		.1018563
12.8	5.670633E-02		.1435139
13.6	6.447648E-02		.1919871
14.4	7.028576E-02		.245892
15.2	.0739274	.3035772	
16	7.541962E-02		.363316
16.8	7.495083E-02		.4234642
17.6	7.281818E-02		.4825718
18.4	6.937155E-02		.5394477
19.2	6.496884E-02		.5931839
20	5.994449E-02		.6431492
20.8	5.459041E-02		.6889632
21.6	4.914688E-02		.7304581
22.4	4.380095E-02		.7676372
23.2	3.868982E-02		.8006335
23.99999	3.390667E-02		.8296721
24.79999	2.950826E-02		.8550381
25.59999	2.552217E-02		.8770502
26.39999	2.195391E-02		.8960407
27.19999	1.879296E-02		.9123395
27.99999	1.601794E-02		.9262638
28.79999	1.360057E-02		.9381112
29.59999	1.150896E-02		.948155
30.39999	9.709841E-03		.9566425
31.19999	8.170269E-03		.9637945
31.99999	6.858709E-03		.9698061
32.79999	5.745843E-03		.974848
33.59999	4.804819E-03		.9790682
34.39999	4.011558E-03		.9825948
35.19999	3.344637E-03		.9855373
35.99999	2.785253E-03		.9879893
36.79998	2.317044E-03		.9900302
37.59998	1.925857E-03		.9917274
38.39998	1.599533E-03		.9931375
39.19998	1.327686E-03		.9943084
39.99998	1.10149E-03	.99528	

The last ordinate of the cumm. dens. func. is set =1

Sorted random values	Distributed random values
3.677011E-04	6.850419
6.756067E-03	8.980216
9.711444E-03	9.331923
2.032906E-02	10.16904
.0237478	10.40104
2.400047E-02	10.41208
2.613592E-02	10.50543
3.551191E-02	10.91527
3.657872E-02	10.9619
4.736662E-02	11.36494
5.589956E-02	11.62845
5.595702E-02	11.63022
5.670512E-02	11.65333
6.105608E-02	11.78769
7.151383E-02	12.08449
7.355577E-02	12.13264
9.629089E-02	12.66876
.1007799	12.77462
.1030828	12.82355
.1064997	12.88917
.1092455	12.9419
.1095739	12.94821
.1100017	12.95643
.1108657	12.97302
.1122803	13.00019
.1145757	13.04427
.1150275	13.05294
.1194881	13.13861
.1201075	13.1505
.1205928	13.15982
.1229543	13.20517
.129147	13.32409
.133105	13.4001
.1421084	13.57301
.1430979	13.59201
.1438367	13.60533
.144884	13.62261
.1485015	13.68231
.156004	13.80614
.1565262	13.81475
.15814	13.84139
.1613985	13.89517
.1675674	13.99698
.1677659	14.00025
.1762406	14.14012
.1771081	14.15444
.17808	14.17048
.1881671	14.33696
.2000077	14.51904
.2004749	14.52597
.2042769	14.58239
.2139184	14.72548
.2190234	14.80125
.2261938	14.90766
.2336911	15.01893
.2394322	15.10413
.2426898	15.15248
.2529784	15.29828
.2652186	15.46803
.2669823	15.49249
.2748073	15.60101
.2853414	15.7471

.2868863	15.76852
.2962654	15.8986
.2973129	15.91312
.300068	15.95133
.3085267	16.06628
.3093378	16.07714
.3096775	16.08169
.3108006	16.09673
.3227947	16.25735
.3277183	16.32329
.328882	16.33887
.3298309	16.35158
.3366159	16.44244
.3395019	16.48109
.3404097	16.49325
.3431954	16.53055
.3515356	16.64224
.3587979	16.73949
.3597357	16.75205
.3609428	16.76822
.3713473	16.90682
.3747999	16.95274
.3797226	17.01822
.3887324	17.13805
.3893401	17.14613
.3899929	17.15482
.393975	17.20778
.4001461	17.28986
.4002991	17.29189
.4082345	17.39744
.4134594	17.46693
.4244857	17.61382
.4310312	17.70242
.4325348	17.72277
.4326887	17.72485
.4331887	17.73162
.4338114	17.74005
.4422875	17.85477
.4441276	17.87967
.4604465	18.10054
.4670625	18.19009
.4694337	18.22218
.4748311	18.29523
.4753552	18.30233
.4793307	18.35613
.4816198	18.38712
.496199	18.59168
.4963169	18.59333
.4982019	18.61985
.5062703	18.73334
.5078771	18.75594
.5156872	18.86579
.5243073	18.98704
.5271801	19.02745
.5273932	19.03045
.5274349	19.03103
.5309483	19.08045
.531449	19.08749
.5378608	19.17768
.5417159	19.23377
.543805	19.26487
.5450828	19.28389
.5497979	19.35409
.5513691	19.37748
.5585553	19.48447
.5611466	19.52304

.5613249	19.5257
.5675404	19.61823
.5694048	19.64599
.5770086	19.75919
.5874845	19.91515
.6018766	20.13918
.6034055	20.16366
.6045134	20.1814
.6054566	20.1965
.6212999	20.45017
.6278098	20.5544
.6321052	20.62317
.6335627	20.64651
.6424253	20.78841
.6429051	20.79609
.6486098	20.89535
.6506249	20.93054
.6529394	20.97096
.6757547	21.36936
.6763578	21.37989
.6771341	21.39344
.6844444	21.5211
.6853203	21.53639
.6913298	21.64563
.7107616	22.02026
.712794	22.05944
.7200381	22.19911
.7362378	22.52436
.7432651	22.67557
.7473214	22.76286
.7546886	22.92138
.7669859	23.18599
.7749153	23.37646
.7867112	23.66245
.7886396	23.70921
.7905869	23.75642
.7948812	23.86053
.7948866	23.86067
.7967906	23.90683
.7974609	23.92308
.8026162	24.05462
.8122079	24.31887
.8178547	24.47444
.8267782	24.72028
.8275138	24.74054
.8280919	24.75647
.8317046	24.8641
.8704043	26.15846
.8745503	26.30914
.8798236	26.51683
.8819329	26.60569
.8854808	26.75515
.8864135	26.79444
.8884345	26.87958
.8981191	27.30202
.8982948	27.31064
.9069754	27.73671
.9201763	28.45025
.9227991	28.60094
.9245142	28.69948
.9278255	28.90545
.9286163	28.95885
.9400988	29.75831
.9473783	30.33813
.9512645	30.69309
.9542702	30.97639

.9602815	31.60704
.9758157	33.78344
.990211	37.68522
.9905262	37.83382
.9936305	39.53688
.9944627	40.0217

mean = 18.84253 std dev.= 5.781599

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	EA/ℓ							$-EA/\ell$						
2		$\frac{12EIz}{\ell^3}$					$\frac{6EIz}{\ell^2}$		$\frac{-12EIz}{\ell^3}$					$\frac{6EIz}{\ell^2}$
3			$\frac{12EIy}{\ell^3}$			$\frac{-6EIy}{\ell^2}$				$\frac{-12EIy}{\ell^3}$			$\frac{-6EIy}{\ell^2}$	
4				$k_{4,4}$	$k_{4,5}$						$k_{4,11}$	$k_{4,12}$		
5				$k_{5,4}$	$k_{5,5}$						$k_{5,11}$	$k_{5,12}$		
6			$\frac{-6EIy}{\ell^2}$			$\frac{4EIy}{\ell}$				$\frac{6EIy}{\ell^2}$			$\frac{2EIy}{\ell}$	
7		$\frac{6EIz}{\ell^2}$					$\frac{4EIz}{\ell}$		$\frac{-6EIz}{\ell^2}$					$\frac{2EIz}{\ell}$
8	$-EA/\ell$			Symmetric				EA/ℓ						
9		$\frac{-12EIz}{\ell^3}$					$\frac{-6EIz}{\ell^2}$		$\frac{12EIz}{L^3}$					$\frac{-6EIz}{\ell^2}$
10			$\frac{-12EIy}{\ell^3}$			$\frac{6EIy}{\ell^2}$				$\frac{12EIy}{L^3}$			$\frac{6EIy}{\ell^2}$	
11				$k_{11,4}$	$k_{11,5}$						$k_{11,11}$	$k_{11,12}$		
12				$k_{12,4}$	$k_{12,5}$						$k_{12,11}$	$k_{12,12}$		
13			$\frac{-6EIy}{\ell^2}$			$\frac{2EIy}{\ell}$				$\frac{6EIy}{\ell^2}$			$\frac{4EIy}{L}$	
14		$\frac{6EIz}{\ell^2}$					$\frac{2EIz}{\ell}$		$\frac{-6EIz}{\ell^2}$					$\frac{4EIz}{\ell}$

$$k(4,4) = \frac{Dk}{g}(-\kappa^2 \sinh \kappa\ell) \quad , \quad k(4,11) = -k(4,4)$$

$$k(4,5) = \frac{Dk}{g}(-\kappa(\cosh \kappa\ell - 1)) \quad , \quad k(4,12) = k(4,5)$$

$$k(5,5) = \frac{Dk}{g}(\sin\kappa\ell - \kappa\ell \cosh \kappa\ell)$$

$$k(5,11) = -k(4,5)$$

$$k(5,12) = \frac{Dk}{g}(\kappa\ell - \sinh \kappa\ell)$$

$$k(11,11) = k(4,4)$$

$$k(11,12) = -k(4,5)$$

$$k(12,12) = k(5,5)$$

where

$$D = EI_{\Omega}$$

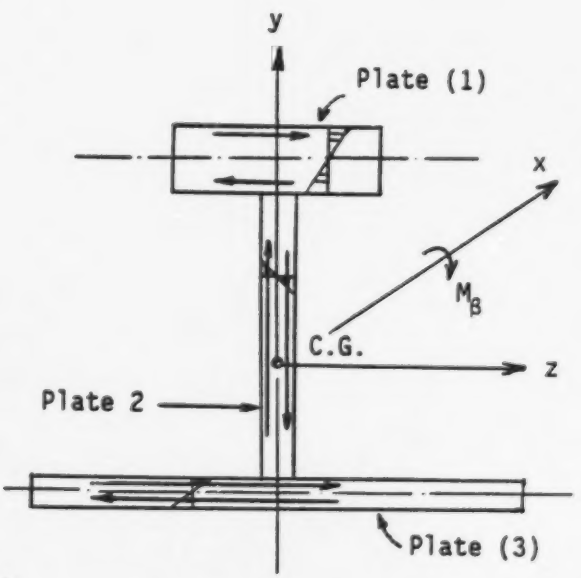
$$g = 2(\cosh \kappa\ell - 1) - \kappa\ell \sinh \kappa\ell$$

Appendix C

EFFECT OF ST. VENANT TORSION ON RAIL BUCKLING

In the following it will be shown that St. Venant torsion (no warping axial stresses) has no effect on the lateral-torsional stability of the rail.

Under St. Venant type torsion a thin-walled section develops a rotary-type shear stress within each plate component such that plates (1) and (3) in Figure 1 develops τ_{zx} & τ_{xz} while plate (2) develops τ_{xy} & τ_{yx}.



It has been shown in 2.3 of Part IV that

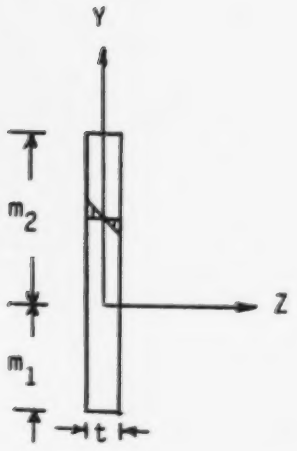
$$I_4 = \int_V 2 \overset{(0)}{\tau}_{xy} (\overset{(2)}{\delta\varepsilon}_{xy}) \, dV$$

where $\overset{(2)}{\delta\varepsilon}_{xy} = \frac{1}{2} \{\delta W_{s,x} \, \delta\beta + (y+e_y) \, \delta\beta_{,x} \, \delta\beta\}$

$\overset{(0)}{\tau}_{xy}$ in the web (plate 2) due to St. Venant torsion is given as

$$\overset{(0)}{\tau}_{xy} (z) = \frac{6}{bt^3} z \overset{2}{M}_\beta (x)$$

where $\overset{2}{M}_\beta(x)$ is the portion of the total M_β carried by the web.



$$I_4 = \frac{6}{bt^3} \int_0^{\ell} \delta W_{s,x} \, \delta\beta \, \overset{2}{M}_\beta(x) \, dx \int_{m_1}^{m_1} \int_{-t/2}^{+t/2} z \, dz \, dy$$

$$+ \frac{6}{bt^3} \int_0^{\ell} \delta W_{s,x} \, \delta\beta \, \overset{2}{M}_\beta(x) \, dx \int_{-m_1}^{m_2} \int_{-t/2}^{+t/2} (zy + z\, e_y) \, dz \, dy$$

since

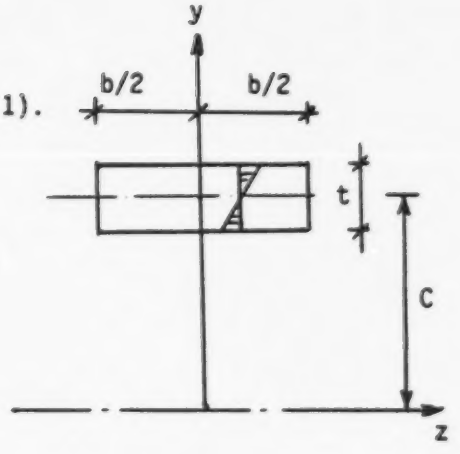
$$\int_{-t/2}^{+t/2} z \, dz = 0.$$

Therefore $I_4 = 0$.

For plate (1)

$$\tau_{xz}(y) = \frac{-6}{bt^3} (y-c) \, \overset{1}{M}_\beta(x)$$

where $\overset{1}{M}_\beta$ is the portion of M_β carried by plate (1).



It has been shown in 2.4 of Part IV that

$$I_5 = \int_V 2 \, \overset{(0)}{\tau}_{xz} \, (\delta \overset{(2)}{\varepsilon}_{xz}) \, dV$$

where

$$\delta \overset{(2)}{\varepsilon}_{xz} = \frac{1}{2} \{-\delta V_{s,x} \, \delta\beta + z \, \delta\beta_{,x} \, \delta\beta\}$$

$$I_5 = \frac{-6}{bt^3} \int_0^{\ell} -\delta V_{s,x} \; \delta\beta \; {}^1M_\beta(x)\, dx \int_{c-t/2}^{c+t/2} \int_{-b/2}^{+b/2} (y-c)\, dz\, dy$$

$$+ \frac{-6}{bt^3} \int_0^{\ell} -\delta V_{s,x} \; \delta\beta \; {}^1M_\beta(x)\, dx \int_{c-t/2}^{c+t/2} \int_{-b/2}^{+b/2} z\,(y-c)\, dz\, dy$$

It can be shown that $\int_{c-t/2}^{c+t/2} (y-c)\, dy = 0.$

and

$$\int_{-b/2}^{+b/2} z\, dz = 0.$$

Therefore $I_5 = 0.$

The same conclusion can be drawn for plate (3). It can thus be concluded that St. Venant torsion has no effect on the lateral-torsional stability of the rail.

Appendix D

Equation (80) in Part IV is detailed in the following. Note that A_{11}, A_{12} $\cdots$, A_{44} are the components of the matrix given in Equation (6) in Part III.

$$
\begin{Bmatrix} \delta C_1 \\ \delta C_2 \\ \delta C_3 \\ \delta C_4 \\ \delta C_5 \\ \delta C_6 \\ \delta C_7 \\ \delta C_8 \\ \delta C_9 \\ \delta C_{10} \\ \delta C_{11} \\ \delta C_{12} \\ \delta C_{13} \\ \delta C_{14} \end{Bmatrix} =
\begin{bmatrix}
-\frac{1}{\ell} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\ell} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{\ell^3} & 0 & 0 & 0 & 0 & \frac{1}{\ell^2} & 0 & \frac{-2}{\ell^3} & 0 & 0 & 0 & 0 & \frac{1}{\ell^2} \\
0 & 0 & \frac{2}{\ell^3} & 0 & 0 & \frac{-1}{\ell^2} & 0 & 0 & 0 & \frac{-2}{\ell^3} & 0 & 0 & \frac{-1}{\ell^2} & 0 \\
0 & 0 & 0 & A_{11} & A_{12} & 0 & 0 & 0 & 0 & 0 & A_{13} & A_{14} & 0 & 0 \\
0 & 0 & 0 & A_{21} & A_{22} & 0 & 0 & 0 & 0 & 0 & A_{23} & A_{24} & 0 & 0 \\
0 & 0 & \frac{-3}{\ell^2} & 0 & 0 & \frac{+2}{\ell} & 0 & 0 & 0 & \frac{+3}{\ell^2} & 0 & 0 & \frac{+1}{\ell} & 0 \\
0 & \frac{-3}{\ell^2} & 0 & 0 & 0 & 0 & \frac{-2}{\ell} & 0 & \frac{+3}{\ell^2} & 0 & 0 & 0 & 0 & \frac{-1}{\ell} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{31} & A_{32} & 0 & 0 & 0 & 0 & 0 & A_{33} & A_{34} & 0 & 0 \\
0 & 0 & 0 & A_{41} & A_{42} & 0 & 0 & 0 & 0 & 0 & A_{43} & A_{44} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix} \delta Uo_p \\ \delta Vs_p \\ \delta Ws_p \\ \delta \beta_p \\ \delta \beta'_p \\ \delta \theta_p \\ \delta \lambda_p \\ \delta Uo_q \\ \delta Vs_q \\ \delta Ws_q \\ \delta \beta_q \\ \delta \beta'_q \\ \delta \theta_q \\ \delta \lambda_q \end{Bmatrix}
$$

Appendix E

A listing of the Fortran Program "RAIL" for stress and stability analysis is given in the following. The program was used to obtain the results in Parts III and IV.

```
      IMPLICIT REAL*8 (A-H,O-Z)                                                 TAR0001C
      REAL*8 TITLE(10)                                                          TAR0002C
      DIMENSION IU(31),IV(31),IW(31),IB(31),IOM(31),IT(31),IL(31)               TAR0003C
     C,SPRING(31,7),AP(31,5),QV(31,5),QH(31,5),ZY(31,5),FIOM(31,5)              TAR0004C
     C,XZ(31,5),XY(31,5),IA(7,31),IIB(7,31),SKM(217,14),GEOMT(217,14)           TAR0005C
     C,SK(14,14),A(14,14,5),C1(14,14),IVC(30,14),SPRINT(31,7,7)                 TAR0006C
     C,TEMP(217,14),UL(217,14),FF(30,14),FVM(31),FQV(31),UF(31,7)               TAR0007C
      DIMENSION PLP(217),U(217),U4(31,7),TFORCE(30,14),F(30,14)                 TAR0008C
     C,TFVM(31),TFQV(31),DFVM(31),DFQV(31),TUF(31,7),AF(14,14)                  TAR0009C
     C,B(217),RM(217),XCO1(31),YCO1(31),YCO(62),XCO(62),KY(30)                  TAR0010C
     C,P(217),TPLP(217),PLPP(217),U1(217)                                       TAR0011C
      READ(5,5001)TITLE                                                         TAR0012C
      READ(5,*)NN,NE,IS,NEMAX,NNMAX,IFLAG,IEXMAX,ISSTMX                         TAR0013C
      IEXX=NN*IS                                                                TAR0014C
      READ(5,*)E,Q                                                              TAR0015C
      READ(5,*)EY,CSTAR,DSTAR,SPACE,IDOSTB                                      TAR0016C
5001  FORMAT(10A8)                                                              TAR0017C
      WRITE(6,6001)TITLE                                                        TAR0018C
6001  FORMAT(' ',10A8)                                                          TAR0019C
      WRITE(6,*)NN,NE,IS,IEX,NEMAX,NNMAX,IFLAG,IEXMAX                           TAR0020C
      WRITE(6,*)E,Q                                                             TAR0021C
      WRITE (6,*)EY,CSTAR,DSTAR,SPACE,IDOSTB                                    TAR0022C
      ISS=2*IS                                                                  TAR0023C
      READ(5,*) ARR,XI,YI,PJ,GI,OMI,OI,B1,GAMA                                  TAR0024C
      WRITE(6,1191)                                                             TAR0025C
      WRITE(6,37)                                                               TAR0026C
37    FORMAT(5X,'THESE ARE THE PROP. OF ELEMENTS',//)                           TAR0027C
      WRITE(6,366) ARR,XI,YI,PJ,GI,OMI,OI,B1,GAMA                               TAR0028C
366    FORMAT(5X,'*',9E12.5)                                                    TAR0029C
      WRITE(6,1191)                                                             TAR0030C
      WRITE(6,4)                                                                TAR0031C
4      FORMAT (5X,'THESE ARE THE PROPERTIES OF THE NODES ',//)                  TAR0032C
      READ(5,*)(IU(I),IV(I),IW(I),IB(I),IOM(I),IT(I),IL(I),I=1,NN)              TAR0033C
      WRITE(6,6)(I,IU(I),IV(I),IW(I),IB(I),IOM(I),IT(I),IL(I),I=1,NN)           TAR0034C
6     FORMAT (5X,I4,'*',7I2)                                                    TAR0035C
      WRITE(6,1191)                                                             TAR0036C
      WRITE(6,101)                                                              TAR0037C
101   FORMAT(5X,'PROPERTIES OF FOUNDATION',//)                                  TAR0038C
      READ(5,*)((SPRING(I,J),J=1,IS),I=1,NN)                                    TAR0039C
      WRITE(6,102)(I,(SPRING(I,J),J=1,IS),I=1,NN)                               TAR0040C
102   FORMAT(5X,I4,'*',7E10.3)                                                  TAR0041C
      WRITE(6,1191)                                                             TAR0042C
      READ(5,*)ECCV,IDIVI,ACCU,ACCS,LIMIT                                       TAR0043C
      WRITE(6,*)ECCV,IDIVI,ACCU,ACCS,LIMIT                                      TAR0044C
      READ(5,*)((AP(I,J),J=1,IDIVI),I=1,NN)                                     TAR0045C
      READ(5,*) NN                                                              TAR0046C
      READ(5,*)((QV(I,J),J=1,IDIVI),I=1,NN)                                     TAR0047C
      READ(5,*) NN                                                              TAR0048C
      READ(5,*)((QH(I,J),J=1,IDIVI),I=1,NN)                                     TAR0049C
      READ(5,*) NN                                                              TAR0050C
      READ(5,*)((ZY(I,J),J=1,IDIVI),I=1,NN)                                     TAR0051C
      READ(5,*) NN                                                              TAR0052C
      READ(5,*)((FIOM(I,J),J=1,IDIVI),I=1,NN)                                   TAR00530
      READ(5,*) NN                                                              TAR00540
      READ(5,*)((XZ(I,J),J=1,IDIVI),I=1,NN)                                     TAR00550
```

```
      READ(5,*) NN                                                          TAR0056C
      READ(5,*)((XY(I,J),J=1,IDIVI),I=1,NN)                                 TAR0057C
      READ(5,*) NN                                                          TAR0058C
      IF(IFLAG.EQ.0)GO TO 904                                               TAR0059C
      READ(5,*)TOTLNG,GSPSG                                                 TAR0060C
904   CONTINUE                                                              TAR0061C
      WRITE(6,211)((AP(I,J),J=1,IDIVI),I=1,NN)                              TAR0062C
      WRITE(6,211)((GV(I,J),J=1,IDIVI),I=1,NN)                              TAR0063C
      WRITE(6,211)((GH(I,J),J=1,IDIVI),I=1,NN)                              TAR0064C
      WRITE(6,211)((ZY(I,J),J=1,IDIVI),I=1,NN)                              TAR0065C
      WRITE(6,211)((FIOM(I,J),J=1,IDIVI),I=1,NN)                            TAR0066C
      WRITE(6,211)((XZ(I,J),J=1,IDIVI),I=1,NN)                              TAR0067C
      WRITE(6,211)((XY(I,J),J=1,IDIVI),I=1,NN)                              TAR0068C
211   FORMAT(5X,5E12.3)                                                     TAR0069C
      WRITE(6,1191)                                                         TAR0070C
      ISTAB=0                                                               TAR0071C
      DO 10 I=1,NN                                                          TAR0072C
      IA(1,I)=IU(I)                                                         TAR0073C
      IA(2,I)=IV(I)                                                         TAR0074C
      IA(3,I)=IW(I)                                                         TAR0075C
      IA(4,I)=IB(I)                                                         TAR0076C
      IA(5,I)=IOM(I)                                                        TAR0077C
      IA(6,I)=IT(I)                                                         TAR0078C
      IA(7,I)=IL(I)                                                         TAR0079C
10    CONTINUE                                                              TAR0080C
      ISUM=0                                                                TAR0081C
      DO 40 I=1,NN                                                          TAR0082C
      DO 40 J=1,IS                                                          TAR0083C
      ISUM=IA(J,I)+ISUM                                                     TAR0084C
      IF(IA(J,I).EQ.0) GO TO 30                                             TAR0085C
      IIB(J,I)=ISUM                                                         TAR0086C
      GO TO 40                                                              TAR0087C
30    IIB(J,I)=0                                                            TAR0088C
40    CONTINUE                                                              TAR0089C
      WRITE(6,13)                                                           TAR0090C
13     FORMAT(5X,'IN THIS MATRIX NON ZERO DIS.  TAKE ASCENDING NO. ',//)    TAR0091C
      DO 104 I=1,NN                                                         TAR0092C
      WRITE(6,14) I,(IIB(J,I),J=1,IS)                                       TAR0093C
14      FORMAT(5X,I5,'*',7I5)                                               TAR0094C
104      CONTINUE                                                           TAR0095C
      IEX=0                                                                 TAR0096C
      DO 156 I=1,IS                                                         TAR0097C
      DO 156 J=1,NN                                                         TAR0098C
      IF(IIB(I,J).GT.IEX)IEX=IIB(I,J)                                       TAR0099C
156   CONTINUE                                                              TAR0100C
      WRITE(6,159) IEX                                                      TAR0101C
159   FORMAT(5X,'DEGREES OF FREEDOM',I5)                                    TAR0102C
      WRITE(6,1191)                                                         TAR0103C
      DO 1222 K=1,NE                                                        TAR0104C
      LL=K                                                                  TAR0105C
      MM=K+1                                                                TAR0106C
      DO 1221 JI=1,IS                                                       TAR0107C
      IVC(K,JI)=IIB(JI,LL)                                                  TAR0108C
      IVC(K,JI+IS)=IIB(JI,MM)                                               TAR0109C
1221  CONTINUE                                                              TAR0110C
```

```
      LLL=MAXO(IVC(K,1),IVC(K,2),IVC(K,3),IVC(K,4),IVC(K,5),IVC(K,6)      TAR0111C
     C,IVC(K,7),IVC(K,8),IVC(K,9),IVC(K,10),IVC(K,11),IVC(K,12)          TAR0112C
     C,IVC(K,13),IVC(K,14))                                              TAR0113C
      DO 1223 I=1,ISS                                                    TAR0114C
      IF(IVC(K,I).EQ.0)IVC(K,I)=99999                                    TAR0115C
1223  CONTINUE                                                           TAR0116C
      MMM=MINO(IVC(K,1),IVC(K,2),IVC(K,3),IVC(K,4),IVC(K,5),IVC(K,6)     TAR0117C
     C,IVC(K,7),IVC(K,8),IVC(K,9),IVC(K,10),IVC(K,11),IVC(K,12)          TAR0118C
     C,IVC(K,13),IVC(K,14))                                              TAR0119C
      IF(MMM.EQ.99999)MMM=0                                              TAR0120C
      KY(K)=IABS(LLL-MMM)                                                TAR0121C
1222  CONTINUE                                                           TAR0122C
      ISST=0                                                             TAR0123C
      DO 1224 K=1,NE                                                     TAR0124C
      IF(KY(K).GT.ISST)ISST=KY(K)                                        TAR0125C
1224  CONTINUE                                                           TAR0126C
      ISST=ISST+1                                                        TAR0127C
      WRITE(6,1225) ISST                                                 TAR0128C
1225  FORMAT(5X,'HALF BAND WIDTH=',I7,//)                                TAR0129C
      DO 60 I=1,IEX                                                      TAR0130C
      DO 60 J=1,ISST                                                     TAR0131C
      SKM(I,J)=0.0                                                       TAR0132C
      GEOMT(I,J)=0.0                                                     TAR0133C
60    CONTINUE                                                           TAR0134C
      CALL BM ELE(G,E,ISS,ARR,XI,YI,PJ,GI,OMI,SK)                        TAR0135C
      IF(IDOSTB.EQ.0)GO TO 501                                           TAR0136C
      CALL GEOM(GI,G,PJ,OMI,E,EY,B1,XI,OI,ARR,A,ISS,GAMA)                TAR0137C
      CALL TRANS(GI,G,E,PJ,OMI,C1,ISS)                                   TAR0138C
501   CONTINUE                                                           TAR0139C
      DO 100 K=1,NE                                                      TAR0140C
       LL=K                                                              TAR0141C
      MM=K+1                                                             TAR0142C
      DO 70  JI=1,IS                                                     TAR0143C
      IVC(K,JI)=IIB(JI,LL)                                               TAR0144C
      IVC(K,JI+IS)=IIB(JI,MM)                                            TAR0145C
70    CONTINUE                                                           TAR0146C
      DO 1207 I=1,ISS                                                    TAR0147C
      IF(IVC(K,I).EQ.0)GO TO 1207                                        TAR0148C
      II=IABS(IVC(K,I))                                                  TAR0149C
      NNZ=II                                                             TAR0150C
      DO 1208 J=I,ISS                                                    TAR0151C
      IF(IVC(K,J).EQ.0)GO TO 1208                                        TAR0152C
      JJ=IABS(IVC(K,J))                                                  TAR0153C
      IF(JJ.LT.II)GO TO 1209                                             TAR0154C
      KKK=JJ                                                             TAR0155C
      LLL=II                                                             TAR0156C
      II=KKK                                                             TAR0157C
      JJ=LLL                                                             TAR0158C
1209  CONTINUE                                                           TAR0159C
      JJ=JJ+(ISST-II)                                                    TAR0160C
      SKM(II,JJ)=SKM(II,JJ)+SK(I,J)                                      TAR0161C
      II=NNZ                                                             TAR0162C
1208  CONTINUE                                                           TAR0163C
1207  CONTINUE                                                           TAR0164C
100    CONTINUE                                                          TAR0165C
```

```
      WRITE(6,15)                                                        TAR0166(
15    FORMAT(5X,'THIS IS THE V-C TABLE',//)                              TAR0167(
      DO 1050  I=1,NE                                                    TAR0168(
      WRITE(6,16)I,(IVC(I,J),J=1,ISS)                                    TAR0169(
16    FORMAT(5X,I5,'*',14I7)                                             TAR0170(
1050    CONTINUE                                                         TAR0171(
      WRITE(6,1191)                                                      TAR0172(
      DO 1304 M=1,NN                                                     TAR0173(
      DO 1305 I=1,IS                                                     TAR0174(
      DO 1305 J=1,IS                                                     TAR0175(
1305  SPRINT(M,I,J)=0.0                                                  TAR0176(
      SPRINT(M,1,1)=SPRING(M,1)                                          TAR0177(
      SPRINT(M,1,7)=DSTAR*SPRING(M,1)                                    TAR0178(
      SPRINT(M,2,2)=SPRING(M,2)                                          TAR0179(
      SPRINT(M,3,3)=SPRING(M,3)                                          TAR0180(
      SPRINT(M,3,4)=-CSTAR*SPRING(M,3)                                   TAR0181(
      SPRINT(M,4,3)=-CSTAR*SPRING(M,3)                                   TAR0182(
      SPRINT(M,4,4)=(CSTAR*CSTAR*SPRING(M,3))+SPRING(M,4)                TAR0183(
      SPRINT(M,5,5)=SPRING(M,5)                                          TAR0184(
      SPRINT(M,6,6)=SPRING(M,6)                                          TAR0185(
      SPRINT(M,7,1)=DSTAR*SPRING(M,1)                                    TAR0186(
      SPRINT(M,7,7)=(DSTAR*DSTAR*SPRING(M,1))+SPRING(M,7)                TAR0187(
1304  CONTINUE                                                           TAR0188(
      DO 1306 M=1,NN                                                     TAR0189(
      DO 1307 I=1,IS                                                     TAR0190(
      IF(IIB(I,M).EQ.0)GO TO 1307                                        TAR0191(
      II=IABS(IIB(I,M))                                                  TAR0192(
      NNZ=II                                                             TAR0193(
      DO 1308 J=I,IS                                                     TAR0194(
      IF(IIB(J,M).EQ.0)GO TO 1308                                        TAR0195(
      JJ=IABS(IIB(J,M))                                                  TAR0196(
      IF(JJ.LT.II)GO TO 1309                                             TAR0197(
      KKK=JJ                                                             TAR0198(
      LLL=II                                                             TAR0199(
      II=KKK                                                             TAR0200(
      JJ=LLL                                                             TAR0201(
1309  CONTINUE                                                           TAR0202(
      JJ=JJ+(ISST-II)                                                    TAR0203(
      SKM(II,JJ)=SKM(II,JJ)+SPRINT(M,I,J)                               TAR0204(
      II=NNZ                                                             TAR0205(
1308  CONTINUE                                                           TAR0206(
1307  CONTINUE                                                           TAR0207(
1306  CONTINUE                                                           TAR0208(
      IF(IDOSTB.EQ.0)GO TO 502                                           TAR0209(
      DO 1004 I=1,IEX                                                    TAR0210(
      DO 1004 J=1,ISST                                                   TAR0211(
      TEMP(I,J)=SKM(I,J)                                                 TAR0212(
1004  CONTINUE                                                           TAR0213(
502   CONTINUE                                                           TAR0214(
      NC=ISST-1                                                          TAR0215(
      CALL LUDAPB(SKM,IEX,NC,IEXMAX,UL,IEXMAX,D1,D2,IER)                 TAR0216(
      DO 105 I=1,NE                                                      TAR0217(
      DO 105 J=1,ISS                                                     TAR0218(
      FF(I,J)=0.0                                                        TAR0219(
105   CONTINUE                                                           TAR0220(
```

```
      DO 204 I=1,NN                                                    TAR0221C
      FVM(I)=0.0                                                       TAR0222C
      FQV(I)=0.0                                                       TAR0223C
      DO 2040 J=1,IS                                                   TAR0224C
      UF(I,J)=0.0                                                      TAR0225C
2040  CONTINUE                                                         TAR0226C
204   CONTINUE                                                         TAR0227C
      DO 6731 I=1,IEXX                                                 TAR0228C
      P(I)=0.0                                                         TAR0229C
6731  CONTINUE                                                         TAR0230C
      DO 1001 JX=1,IDIVI                                               TAR0231C
      WRITE(6,6761) JX                                                 TAR0232C
6761  FORMAT(5X,'THIS IS THE',I4,3X,'LOAD INCREMENT',/)                TAR0233C
      K=0                                                              TAR0234C
      DO 20 I=1,NN                                                     TAR0235C
      PLP(K+1)=AP(I,JX)                                                TAR0236C
      PLP(K+2)=QV(I,JX)                                                TAR0237C
      PLP(K+3)=QH(I,JX)                                                TAR0238C
      PLP(K+4)=ZY(I,JX)                                                TAR0239C
      PLP(K+5)=FIOM(I,JX)                                              TAR0240C
      PLP(K+6)=XZ(I,JX)                                                TAR0241C
      PLP(K+7)=XY(I,JX)                                                TAR0242C
      K=K+IS                                                           TAR0243C
20    CONTINUE                                                         TAR0244C
      K=1                                                              TAR0245C
      DO 150 KK=1,IEXX                                                 TAR0246C
      I=((KK-1)/IS)+1                                                  TAR0247C
      J=KK-(I*IS)+IS                                                   TAR0248C
      IF(IIB(J,I).EQ.0)GO TO 150                                       TAR0249C
      PLPP(K)=PLP(KK)                                                  TAR0250C
      K=K+1                                                            TAR0251C
150   CONTINUE                                                         TAR0252C
      SUM=0.0                                                          TAR0253C
      DO 606 I=1,IEXX                                                  TAR0254C
      SUM=SUM+(PLP(I)**2)                                              TAR0255C
606   CONTINUE                                                         TAR0256C
      CONV=SUM*ACCU                                                    TAR0257C
      CALL LUELPB(UL,PLPP,IEX,NC,IEXMAX,U1)                            TAR0258C
      K=1                                                              TAR0259C
      DO 153 I=1,NN                                                    TAR0260C
      DO 153 J=1,IS                                                    TAR0261C
      KK=J+(I*IS)-IS                                                   TAR0262C
      IF(IIB(J,I).EQ.0)GO TO 154                                       TAR0263C
      U(KK)=U1(K)                                                      TAR0264C
      K=K+1                                                            TAR0265C
      GO TO 153                                                        TAR0266C
154   U(KK)=0.0                                                        TAR0267C
153   CONTINUE                                                         TAR0268C
      K=0                                                              TAR0269C
      DO 260 I=1,NN                                                    TAR0270C
      DO 250 J=1,IS                                                    TAR0271C
      U4(I,J)=U(K+J)                                                   TAR0272C
250   CONTINUE                                                         TAR0273C
      K=K+IS                                                           TAR0274C
260   CONTINUE                                                         TAR0275C
```

```
      WRITE(6,1191)                                                    TAR0276(
      DO 280 I=1,NE                                                    TAR0277(
      DO 106 J=1,ISS                                                   TAR0278(
      TFORCE(I,J)=FF(I,J)                                              TAR0279(
106   CONTINUE                                                         TAR0280(
      ICODE=0                                                          TAR0281(
      CALL FORCE(SK,U4,I,ISS,NE,NEMAX,F,IS,NNMAX,ICODE)                TAR0282(
280   CONTINUE                                                         TAR0283(
      DO 290 I=1,NN                                                    TAR0284(
      TFVM(I)=FVM(I)                                                   TAR0285(
      TFQV(I)=FQV(I)                                                   TAR0286(
      DFVM(I)=(SPRING(I,2)*U4(I,2))                                    TAR0287(
      DFQV(I)=QV(I,JX)                                                 TAR0288(
      DO 2903 J=1,IS                                                   TAR0289(
      TUF(I,J)=UF(I,J)                                                 TAR0290(
2903  CONTINUE                                                         TAR0291(
290   CONTINUE                                                         TAR0292(
      DO 6721 I=1,IEXX                                                 TAR0293(
      TPLP(I)=P(I)                                                     TAR0294(
6721  CONTINUE                                                         TAR0295(
      FACT=1.0                                                         TAR0296(
1092  CONTINUE                                                         TAR0297(
      DO 1115 I=1,NE                                                   TAR0298(
      DO 107 J=1,ISS                                                   TAR0299(
      F(I,J)=F(I,J)*FACT                                               TAR0300(
      FF(I,J)=TFORCE(I,J)+F(I,J)                                       TAR0301(
107   CONTINUE                                                         TAR0302(
      IF(IDOSTB.EQ.0)GO TO 503                                         TAR0303(
      CALL GEMF(I,FF,NEMAX,ISS,A,AF)                                   TAR0304(
      CALL GLGEMF(C1,AF,ISS,I)                                         TAR0305(
      DO 1107 KK=1,ISS                                                 TAR0306(
      IF(IVC(I,KK).EQ.0)GO TO 1107                                     TAR0307(
      II=IABS(IVC(I,KK))                                               TAR0308(
      NNZ=II                                                           TAR0309(
      DO 1108 J=KK,ISS                                                 TAR0310(
      IF(IVC(I,J).EQ.0)GO TO 1108                                      TAR0311(
      JJ=IABS(IVC(I,J))                                                TAR0312(
      IF(JJ.LT.II)GO TO 1109                                           TAR0313(
      KKK=JJ                                                           TAR0314(
      LLL=II                                                           TAR0315(
      II=KKK                                                           TAR0316(
      JJ=LLL                                                           TAR0317(
1109  CONTINUE                                                         TAR0318(
      JJ=JJ+(ISST-II)                                                  TAR0319(
      GEOMT(II,JJ)=GEOMT(II,JJ)+AF(KK,J)                               TAR0320(
      II=NNZ                                                           TAR0321(
1108  CONTINUE                                                         TAR0322(
1107  CONTINUE                                                         TAR0323(
503   CONTINUE                                                         TAR0324(
1115  CONTINUE                                                         TAR0325(
      DO 6741 I=1,IEXX                                                 TAR0326(
      PLP(I)=PLP(I)*FACT                                               TAR0327(
      P(I)=TPLP(I)+PLP(I)                                              TAR0328(
6741  CONTINUE                                                         TAR0329(
      DO 36 I=1,NN                                                     TAR0330(
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      DFVM(I)=DFVM(I)*FACT                                                  TAR03310
      FVM(I)=TFVM(I)+DFVM(I)                                                TAR03320
      DFQV(I)=DFQV(I)*FACT                                                  TAR03330
      FQV(I)=TFQV(I)+DFQV(I)                                                TAR03340
      IF(IDOSTB.EQ.0)GO TO 504                                              TAR03350
      MM=IIB(4,I)                                                           TAR03360
      IF(MM.EQ.0)GO TO 504                                                  TAR03370
      GEOMT(MM,ISST)=GEOMT(MM,ISST)-(FVM(I)*(-DSTAR+EY))                    TAR03380
     C+(FQV(I)*ECCV)                                                        TAR03390
504   CONTINUE                                                              TAR03400
      DO 362 J=1,IS                                                         TAR03410
      U4(I,J)=U4(I,J)*FACT                                                  TAR03420
      UF(I,J)=TUF(I,J)+U4(I,J)                                              TAR03430
362   CONTINUE                                                              TAR03440
36    CONTINUE                                                              TAR03450
      WRITE(6,193)                                                          TAR03460
193    FORMAT(2X,'MATRIX ,REAL DISPLACEMENT INCREMENT OF NODES',//)         TAR03470
      DO 113 I=1,NN                                                         TAR03480
      WRITE(6,34)I,(U4(I,J),J=1,IS)                                         TAR03490
34      FORMAT(5X,I4,'*',7E15.5)                                            TAR03500
113     CONTINUE                                                            TAR03510
      WRITE(6,508)                                                          TAR03520
508   FORMAT(5X,'LOAD INCREMENT AND TOTAL LOADS REACHED SO FAR',//)         TAR03530
      SUM=0.0                                                               TAR03540
      DO 505 I=1,IEXX                                                       TAR03550
      SUM=SUM+(PLP(I)**2)                                                   TAR03560
      WRITE(6,506) I,PLP(I),P(I)                                            TAR03570
505   CONTINUE                                                              TAR03580
      IF(SUM.LE.CONV)GO TO 1000                                             TAR03590
506   FORMAT(5X,I4,'*',3X,E20.10,3X,E20.10)                                 TAR03600
      IF(IDOSTB.EQ.0)GO TO 1001                                             TAR03610
      DO 1020 I=1,IEX                                                       TAR03620
      DO 1020 J=1,ISST                                                      TAR03630
      GEOMT(I,J)=TEMP(I,J)+GEOMT(I,J)                                       TAR03640
1020  CONTINUE                                                              TAR03650
      XD=0.0                                                                TAR03660
      CALL STABC(GEOMT,B,RM,XD,ACCS,IEXMAX,ISS,IEX,ISST,LIMIT)              TAR03670
      DO 24 I=1,IEX                                                         TAR03680
      DO 24 J=1,ISST                                                        TAR03690
24    GEOMT(I,J)=0.0                                                        TAR03700
      IF(XD.LE.0.0)ISTAB=1                                                  TAR03710
      IF(XD.GT.0.0)GO TO 2152                                               TAR03720
      IF(ISTAB.EQ.1)FACT=1./2.                                              TAR03730
      IF(ISTAB.EQ.1)GO TO 1092                                              TAR03740
2152  CONTINUE                                                              TAR03750
      IF(ISTAB.EQ.0)GO TO 1001                                              TAR03760
      DO 1016 I=1,NE                                                        TAR03770
      DO 1016 J=1,ISS                                                       TAR03780
1016  TFORCE(I,J)=FF(I,J)                                                   TAR03790
      DO 6751 I=1,IEXX                                                      TAR03800
      TPLP(I)=P(I)                                                          TAR03810
6751  CONTINUE                                                              TAR03820
      DO 1017 I=1,NN                                                        TAR03830
      TFVM(I)=FVM(I)                                                        TAR03840
      TFQV(I)=FQV(I)                                                        TAR03850
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      DO 1018 J=1,IS                                                    TAR0386C
1018  TUF(I,J)=UF(I,J)                                                  TAR0387C
1017  CONTINUE                                                          TAR0388C
      IF(ISTAB.EQ.1)FACT=1./2.                                          TAR0389C
      IF(ISTAB.EQ.1)GO TO 1092                                          TAR0390C
1001    CONTINUE                                                        TAR0391C
1000  CONTINUE                                                          TAR0392C
      DO 6781 I=1,IEXX                                                  TAR0393C
      WRITE(6,6791) I,P(I)                                              TAR0394C
6791  FORMAT(5X,'MAXIMUM LOADS',I5,'*',F20.10)                          TAR0395C
6781  CONTINUE                                                          TAR0396C
      WRITE(6,6891)                                                     TAR0397C
      DO 9854 I=1,NN                                                    TAR0398C
      WRITE(6,34)I,  (UF(I,J),J=1,IS)                                   TAR0399C
9854  CONTINUE                                                          TAR0400C
6891  FORMAT(5X,'PRE-BUCKLING DISPLACEMENTS',//)                        TAR0401C
      WRITE(6,1191)                                                     TAR0402C
      DO 7031 I=1,NE                                                    TAR0403C
      ICODE=1                                                           TAR0404C
      CALL FORCE(SK,UF,I,ISS,NE,NEMAX,F,IS,NNMAX,ICODE)                 TAR0405C
7031  CONTINUE                                                          TAR0406C
      CALL FOUNDT(SPRING,NN,UF,CSTAR,DSTAR,EY,IS,SPRINT,NNMAX)          TAR0407C
1191  FORMAT(2X,'**************************************************')  TAR0408C
      IF(IDOSTB.EQ.0)GO TO 507                                          TAR0409C
      DO 6771 I=1,IEX                                                   TAR0410C
      WRITE(6,6778)I,RM(I)                                              TAR0411C
6771  CONTINUE                                                          TAR0412C
6778  FORMAT(5X,'BUCKLING MODE',3X,I6,'*',E20.10)                       TAR0413C
507   CONTINUE                                                          TAR0414C
      IF(IFLAG.EQ.0)GO TO 72                                            TAR0415C
      DO 103 I=1,NN                                                     TAR0416C
      UF(I,3)=UF(I,3)+(UF(I,4)*EY)                                      TAR0417C
103   CONTINUE                                                          TAR0418C
      DO 1060 I=1,NE                                                    TAR0419C
      F(I,4)=F(I,4)-(F(I,3)*EY)                                         TAR0420C
      F(I,11)=F(I,11)-(F(I,10)*EY)                                      TAR0421C
1060   CONTINUE                                                         TAR0422C
      CALL IMP                                                          TAR0423C
      CALL NOBRDR                                                       TAR0424C
      CALL YNMADJ('ENDS')                                               TAR0425C
      CALL PHYSOR(2.,9.7)                                               TAR0426C
      CALL AREA2D (4.,  .5)                                             TAR0427C
      CALL HEADIN('Displacements Along Rail Centroid$',100,1.9,2)       TAR0428C
      CALL HEADIN('(in inches ,radians and radians per inch)$',100,1.,2)TAR0429C
      CALL XNAME(' ',1)                                                 TAR0430C
      XVAL=XMESS('distance in inches$',100)                             TAR0431C
      CALL HEIGHT(.1)                                                   TAR0432C
      CALL MESSAG('distance in inches$',100,((4.-XVAL)/2.)+.25,.20)     TAR0433C
      CALL RESET('HEIGHT')                                              TAR0434C
      CALL GRAF(0.,GSPSG,TOTLNG,  0.,1.,1.)                             TAR0435C
      CALL ENDGR(0)                                                     TAR0436C
      CALL PHYSOR(2.,8.2)                                               TAR0437C
      DO 51 J=1,IS                                                      TAR0438C
      DO 52 I=1,NN                                                      TAR0439C
      XCO1(I)=(I-1)*SPACE                                               TAR0440C
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      YCO1(I)=UF(I,J)                                                    TAR0441C
52    CONTINUE                                                           TAR0442C
      YMIN=YCO1(1)                                                       TAR0443C
      YMAX=YCO1(1)                                                       TAR0444C
      DO 53 I=2,NN                                                       TAR0445C
      IF(YCO1(I).GT.YMAX)YMAX=YCO1(I)                                    TAR0446C
      IF(YCO1(I).LT.YMIN)YMIN=YCO1(I)                                    TAR0447C
53    CONTINUE                                                           TAR0448C
      CALL PLOT1(YMIN,YMAX,XCO1,YCO1,NN,J,TOTLNG,GSPSG)                  TAR0449C
      CALL OREL(0.,-1.3)                                                 TAR0450C
51    CONTINUE                                                           TAR0451C
      CALL ENDPL (1)                                                     TAR0452C
      CALL NOBRDR                                                        TAR0453C
      CALL RESET('XNONUM')                                               TAR0454C
      CALL RESET('XTICKS')                                               TAR0455C
      CALL RESET('XNAME')                                                TAR0456C
      CALL PHYSOR(2.,9.7)                                                TAR0457C
      CALL AREA2D (4., .5)                                               TAR0458C
      CALL HEADIN('Internal Forces Along Rail Centroid$',100,1.9,2)      TAR0459C
      CALL HEADIN('(in Kips,Kips.inch and Kips.inch.inch)$',100,1.,2)    TAR0460C
      CALL XNAME(' ',1)                                                  TAR0461C
      CALL RESET('YNAME')                                                TAR0462C
      XVAL=XMESS('distance in inches$',100)                              TAR0463C
      CALL HEIGHT(.1)                                                    TAR0464C
      CALL MESSAG('distance in inches$',100,((4.-XVAL)/2.)+.25,.20)      TAR0465C
      CALL RESET('HEIGHT')                                               TAR0466C
      CALL GRAF(0.,GSPSG,TOTLNG, 0.,1.,1.)                              TAR0467C
      CALL ENDGR(0)                                                      TAR0468C
      CALL PHYSOR(2.,8.2)                                                TAR0469C
      DO 56 J=1,IS                                                       TAR0470C
      K=0                                                                TAR0471C
      IF(J.EQ.1)KKK=1                                                    TAR0472C
      IF(J.EQ.2)KKK=-1                                                   TAR0473C
      IF(J.EQ.3)KKK=1                                                    TAR0474C
      IF(J.EQ.4)KKK=-1                                                   TAR0475C
      IF(J.EQ.5)KKK=1                                                    TAR0476C
      IF(J.EQ.6)KKK=1                                                    TAR0477C
      IF(J.EQ.7)KKK=1                                                    TAR0478C
      DO 57 I=1,NN                                                       TAR0479C
      IF(I.EQ.1)YCO(I+K)=KKK*(-F(I,J))                                   TAR0480C
      IF(I.EQ.1)YCO(I+1+K)=KKK*(-F(I,J))                                 TAR0481C
      IF(I.EQ.1)GO TO 58                                                 TAR0482C
      IF(I.EQ.NN)YCO(I+K)=KKK*(F(I-1,J+IS))                              TAR0483C
      IF(I.EQ.NN)YCO(I+1+K)=KKK*(F(I-1,J+IS))                            TAR0484C
      IF(I.EQ.NN)GO TO 58                                                TAR0485C
      YCO(I+K)=KKK*F(I-1,J+IS)                                           TAR0486C
      YCO(I+1+K)=KKK*(-F(I,J))                                           TAR0487C
58    CONTINUE                                                           TAR0488C
      K=K+1                                                              TAR0489C
57    CONTINUE                                                           TAR0490C
      NNN=2*NN                                                           TAR0491C
      YMIN=YCO(1)                                                        TAR0492C
      YMAX=YCO(1)                                                        TAR0493C
      DO 59 I=2,NNN                                                      TAR0494C
      IF(YCO(I).GT.YMAX)YMAX=YCO(I)                                      TAR0495C
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      IF(YCO(I).LT.YMIN)YMIN=YCO(I)                                      TAR0496(
59    CONTINUE                                                           TAR0497(
      K=0                                                                TAR0498(
      DO 55 I=1,NN                                                       TAR0499(
      XCO(I+K)=(I-1)*SPACE                                               TAR0500(
      XCO(I+1+K)=(I-1)*SPACE                                             TAR0501(
      K=K+1                                                              TAR0502(
55    CONTINUE                                                           TAR0503(
      CALL PLOT2(YMIN,YMAX,XCO,YCO,NNN,J,TOTLNG,GSPSG)                   TAR0504(
      CALL OREL(0.,-1.3)                                                 TAR0505(
56    CONTINUE                                                           TAR0506(
      CALL ENDPL (2)                                                     TAR0507(
      IF(IDOSTB.EQ.0)CALL DONEPL                                         TAR0508(
      IF(IDOSTB.EQ.0)GO TO 72                                            TAR0509(
      CALL NOBRDR                                                        TAR0510(
      CALL RESET('XNONUM')                                               TAR0511(
      CALL RESET('XTICKS')                                               TAR0512(
      CALL RESET('XNAME')                                                TAR0513(
      CALL PHYSOR(2.,9.7)                                                TAR0514(
      CALL AREA2D(4.,.5)                                                 TAR0515(
      CALL HEADIN('Buckling Modes Along Rail Shear Centre$',100,1.7,2)   TAR0516(
      CALL HEADIN('(Dimensionless)$',100,1.,2)                           TAR0517(
      CALL XNAME(' ',1)                                                  TAR0518(
      CALL RESET('YNAME')                                                TAR0519(
      XVAL=XMESS('distance in inches$',100)                              TAR0520(
      CALL HEIGHT(.1)                                                    TAR0521(
      CALL MESSAG('distance in inches$',100,((4.-XVAL)/2.)+.25,.20)      TAR0522(
      CALL RESET('HEIGHT')                                               TAR0523(
      CALL GRAF(0.,GSPSG,TOTLNG,0.,1.,1.)                                TAR0524(
      CALL ENDGR(0)                                                      TAR0525(
      CALL PHYSOR(2.,8.2)                                                TAR0526(
      DO 5111 J=1,IS                                                     TAR0527(
      DO 5211 I=1,NN                                                     TAR0528(
      XCO1(I)=(I-1)*SPACE                                                TAR0529(
      MMM=IIB(J,I)                                                       TAR0530(
      IF(MMM.EQ.0)YCO1(I)=0.0                                            TAR0531(
      IF(MMM.NE.0)YCO1(I)=RM(MMM)                                        TAR0532(
5211  CONTINUE                                                           TAR0533(
      YMIN=YCO1(1)                                                       TAR0534(
      YMAX=YCO1(1)                                                       TAR0535(
      DO 5311 I=2,NN                                                     TAR0536(
      IF(YCO1(I).GT.YMAX)YMAX=YCO1(I)                                    TAR0537(
      IF(YCO1(I).LT.YMIN)YMIN=YCO1(I)                                    TAR0538(
5311  CONTINUE                                                           TAR0539(
      CALL PLOT3(YMIN,YMAX,XCO1,YCO1,NN,J,TOTLNG,GSPSG)                  TAR0540(
      CALL OREL(0.,-1.3)                                                 TAR0541(
5111  CONTINUE                                                           TAR0542(
      CALL ENDPL(3)                                                      TAR0543(
      CALL DONEPL                                                        TAR0544(
72    CONTINUE                                                           TAR0545(
      STOP                                                               TAR0546(
      END                                                                TAR0547(
      SUBROUTINE BM ELE(Q,E,ISS,ARR,XI,YI,PJ,SL,OMG,SK)                  TAR0548(
      IMPLICIT REAL*8(A-H,O-Z)                                           TAR0549(
      DIMENSION SK(ISS,ISS)                                              TAR0550(
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      DO 10 I=1,ISS                                                    TAR0551(
      DO 10 J=1,ISS                                                    TAR0552(
      SK(I,J)=0.0                                                      TAR0553(
10    CONTINUE                                                         TAR0554(
      SK(1,1)=(E*ARR)/SL                                               TAR0555(
      SK(1,8)=-SK(1,1)                                                 TAR0556(
      SK(2,2)=(12.*E*XI)/(SL**3)                                       TAR0557(
      SK(2,7)=(6.*E*XI)/(SL**2)                                        TAR0558(
      SK(2,9)=-SK(2,2)                                                 TAR0559(
      SK(2,14)=SK(2,7)                                                 TAR0560(
      SK(3,3)=(12.*E*YI)/(SL**3)                                       TAR0561(
      SK(3,6)=(-6.*E*YI)/(SL**2)                                       TAR0562(
      SK(3,10)=-SK(3,3)                                                TAR0563(
      SK(3,13)=SK(3,6)                                                 TAR0564(
      SK(6,6)=(4.*E*YI)/SL                                             TAR0565(
      SK(6,10)=(6.*E*YI)/(SL**2)                                       TAR0566(
      SK(6,13)=SK(6,6)/2.                                              TAR0567(
      SK(7,7)=(4.*E*XI)/SL                                             TAR0568(
      SK(7,9)=(-6.*E*XI)/(SL**2)                                       TAR0569(
      SK(7,14)=SK(7,7)/2.                                              TAR0570(
      SK(8,8)=SK(1,1)                                                  TAR0571(
      SK(9,9)=SK(2,2)                                                  TAR0572(
      SK(9,14)=-SK(2,14)                                               TAR0573(
      SK(10,10)=SK(3,3)                                                TAR0574(
      SK(10,13)=-SK(3,13)                                              TAR0575(
      SK(13,13)=SK(6,6)                                                TAR0576(
      SK(14,14)=SK(7,7)                                                TAR0577(
      FK=DSQRT((G*PJ)/(E*OMG))                                         TAR0578(
      F=FK*SL                                                          TAR0579(
      C=DCOSH(F)                                                       TAR0580(
      S=DSINH(F)                                                       TAR0581(
      GG=(2.*(C-1.))-(F*S)                                             TAR0582(
      D=E*OMG                                                          TAR0583(
      UML=(D*FK)/GG                                                    TAR0584(
      SK(4,4)=(-FK*FK*S)*UML                                           TAR0585(
      SK(4,5)=(-FK*(C-1.))*UML                                         TAR0586(
      SK(4,11)=-SK(4,4)                                                TAR0587(
      SK(4,12)=SK(4,5)                                                 TAR0588(
      SK(5,5)=(((F*C)-S)*(-1.))*UML                                    TAR0589(
      SK(5,11)=(FK*(C-1.))*UML                                         TAR0590(
      SK(5,12)=(F-S)*UML                                               TAR0591(
      SK(11,11)=SK(4,4)                                                TAR0592(
      SK(11,12)=-SK(4,5)                                               TAR0593(
      SK(12,12)=SK(5,5)                                                TAR0594(
      DO 20 I=1,ISS                                                    TAR0595(
      DO 20 J=I,ISS                                                    TAR0596(
      SK(J,I)=SK(I,J)                                                  TAR0597(
20    CONTINUE                                                         TAR0598(
      RETURN                                                           TAR0599(
      END                                                              TAR0600(
C                                                                      TAR0601(
C                                                                      TAR0602(
      SUBROUTINE  FORCE(SK,U4,I,ISS,NE,NEMAX,F,IS,NN,ICODE)            TAR0603(
      IMPLICIT REAL*8(A-H,O-Z)                                         TAR0604(
      DIMENSION SK(ISS,ISS),U4(NN,IS),F(NEMAX,ISS),U5(14),U3(14)       TAR0605(
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      IF(ICODE.EQ.0)GO TO 55                                                    TAR0606(
      WRITE(6,1903)I                                                            TAR0607(
1903  FORMAT(5X,'END FORCES ACTING ON ELEMENT NO.(',I2,')',/)                   TAR0608(
55    CONTINUE                                                                  TAR0609(
      DO 5 K=1,IS                                                               TAR0610(
      U5(K)=U4(I,K)                                                             TAR0611(
      U5(K+IS)=U4(I+1,K)                                                        TAR0612(
5     CONTINUE                                                                  TAR0613(
      DO 10 K=1,ISS                                                             TAR0614(
      U3(K)=0.0                                                                 TAR0615(
      DO 10 J=1,ISS                                                             TAR0616(
      U3(K)=U3(K)+(U5(J)*SK(K,J))                                               TAR0617(
10    CONTINUE                                                                  TAR0618(
      DO 20 K=1,ISS                                                             TAR0619(
      F(I,K)=U3(K)                                                              TAR0620(
      IF(ICODE.EQ.0)GO TO 66                                                    TAR0621(
      WRITE(6,35) K,F(I,K)                                                      TAR0622(
35    FORMAT(5X,I4,'*',E20.12)                                                  TAR0623(
66    CONTINUE                                                                  TAR0624(
20    CONTINUE                                                                  TAR0625(
      RETURN                                                                    TAR0626(
      END                                                                       TAR0627(
C                                                                               TAR0628(
C                                                                               TAR0629(
      SUBROUTINE FOUNDT(B,NN,U,CT,DT,EY,IS,S,NNMAX)                             TAR0630(
      IMPLICIT REAL*8 (A-H,O-Z)                                                 TAR0631(
      DIMENSION B(NNMAX,IS),U(NNMAX,IS),F(7),S(NNMAX,IS,IS),T(7,7),UU(7)        TAR0632(
      DO 10 I=1,NN                                                              TAR0633(
      DO 20 J=1,IS                                                              TAR0634(
      F(J)=0.0                                                                  TAR0635(
      DO 30 K=1,IS                                                              TAR0636(
      F(J)=F(J)+(S(I,J,K)*U(I,K))                                               TAR0637(
30    CONTINUE                                                                  TAR0638(
20    CONTINUE                                                                  TAR0639(
      WRITE(6,1192)I                                                            TAR0640(
1192  FORMAT(5X,'GLOBAL FORCES IN SPRING',3X,I6)                                TAR0641(
      WRITE(6,1193)(F(J),J=1,7)                                                 TAR0642(
1193  FORMAT(5X,7E15.7)                                                         TAR0643(
      DO 40 J=1,IS                                                              TAR0644(
      DO 50 K=1,IS                                                              TAR0645(
50    T(J,K)=0.0                                                                TAR0646(
      T(J,J)=1.0                                                                TAR0647(
40    CONTINUE                                                                  TAR0648(
      T(1,7)=DT                                                                 TAR0649(
      T(3,4)=-CT                                                                TAR0650(
      DO 60 J=1,IS                                                              TAR0651(
      UU(J)=0.0                                                                 TAR0652(
      DO 70 K=1,IS                                                              TAR0653(
      UU(J)=UU(J)+(T(J,K)*U(I,K))                                               TAR0654(
70    CONTINUE                                                                  TAR0655(
      F(J)=UU(J)*B(I,J)                                                         TAR0656(
60    CONTINUE                                                                  TAR0657(
      WRITE(6,1194)(F(J),J=1,7)                                                 TAR0658(
1194  FORMAT(3X,'ACTUAL SPRING FORCES',3X,7E15.7)                               TAR0659(
      WRITE(6,1191)                                                             TAR0660(
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1191  FORMAT(2X,'****************************************')          TAR0661(
10    CONTINUE                                                      TAR0662(
      RETURN                                                        TAR0663(
      END                                                           TAR0664(
C                                                                   TAR0665(
C                                                                   TAR0666(
      SUBROUTINE PLOT1(YMIN,YMAX,X,Y,N,J,TOTLNG,GSPSG)              TAR0667(
      IMPLICIT REAL*8(A-H,O-Z)                                      TAR0668(
      REAL*4    X(1),Y(1)                                           TAR0669(
      DO 10 I=1,N                                                   TAR0670(
          X(I) = X(2*I-1)                                           TAR0671(
          Y(I) = Y(2*I-1)                                           TAR0672(
   10 CONTINUE                                                      TAR0673(
      CALL AREA2D(4.,.9)                                            TAR0674(
      CALL XNAME(' ',1)                                             TAR0675(
      IF(J.EQ.1)CALL YNAME('U$',100)                                TAR0676(
      IF(J.EQ.2)CALL YNAME('V$',100)                                TAR0677(
      IF(J.EQ.3)CALL YNAME('W$',100)                                TAR0678(
      IF(J.EQ.4)CALL YNAME('BETA$',100)                             TAR0679(
      IF(J.EQ.5)CALL YNAME('OMEGA$',100)                            TAR0680(
      IF(J.EQ.6)CALL YNAME('THETA$',100)                            TAR0681(
      IF(J.EQ.7)CALL YNAME('LAMDA$',100)                            TAR0682(
      CALL XNONUM                                                   TAR0683(
      CALL YAXANG(0.)                                               TAR0684(
      CALL XTICKS(5)                                                TAR0685(
      CALL CROSS                                                    TAR0686(
      IF(YMIN.EQ.0. .AND.YMAX.EQ.0.)THEN                            TAR0687(
      YMIN=-1.                                                      TAR0688(
      YMAX=+1.                                                      TAR0689(
      ENDIF                                                         TAR0690(
      IF(YMIN.LT.0. .AND.YMAX.LT.0.)YMAX=0.                         TAR0691(
      IF(YMIN.GT.0. .AND.YMAX.GT.0.)YMIN=0.                         TAR0692(
      CALL GRAF(0.,GSPSG,TOTLNG,YMIN,'SCALE',YMAX)                  TAR0693(
      CALL SCLPIC (.25)                                             TAR0694(
      CALL MARKER(15)                                               TAR0695(
      CALL PSPLIN                                                   TAR0696(
      CALL CURVE(X,Y,N,1)                                           TAR0697(
      CALL ENDGR (0)                                                TAR0698(
      RETURN                                                        TAR0699(
      END                                                           TAR0700(
C                                                                   TAR0701(
C                                                                   TAR0702(
      SUBROUTINE PLOT2(YMIN,YMAX,X,Y,N,J,TOTLNG,GSPSG)              TAR0703(
      IMPLICIT REAL*8(A-H,O-Z)                                      TAR0704(
      REAL*4    X(1),Y(1)                                           TAR0705(
      DO 10 I=1,N                                                   TAR0706(
          X(I) = X(2*I-1)                                           TAR0707(
          Y(I) = Y(2*I-1)                                           TAR0708(
   10 CONTINUE                                                      TAR0709(
      CALL AREA2D(4.,.9)                                            TAR0710(
      CALL XNAME(' ',1)                                             TAR0711(
      IF(J.EQ.1)CALL YNAME('FU$',100)                               TAR0712(
      IF(J.EQ.2)CALL YNAME('FV$',100)                               TAR0713(
      IF(J.EQ.3)CALL YNAME('FW$',100)                               TAR0714(
      IF(J.EQ.4)CALL YNAME('M BETA$',100)                           TAR0715(
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      IF(J.EQ.5)CALL YNAME('M OMEGA$',100)                              TAR0716(
      IF(J.EQ.6)CALL YNAME('M THETA$',100)                              TAR0717(
      IF(J.EQ.7)CALL YNAME('M LAMDA$',100)                              TAR0718(
      CALL XNONUM                                                       TAR0719(
      CALL YAXANG(0.)                                                   TAR0720(
      CALL XTICKS(5)                                                    TAR0721(
      CALL CROSS                                                        TAR0722(
      IF(YMIN.EQ.0. .AND. YMAX.EQ.0.)THEN                               TAR0723(
      YMIN=-1.                                                          TAR0724(
      YMAX=+1.                                                          TAR0725(
      ENDIF                                                             TAR0726(
      IF(YMIN.LT.0. .AND. YMAX.LT.0.)YMAX=0.                            TAR0727(
      IF(YMIN.GT.0. .AND. YMAX.GT.0.)YMIN=0.                            TAR0728(
      CALL GRAF(0.,GSPSG,TOTLNG,YMIN,'SCALE',YMAX)                      TAR0729(
      CALL SCLPIC (.25)                                                 TAR0730(
      CALL MARKER(15)                                                   TAR0731(
      CALL LINEAR                                                       TAR0732(
      CALL CURVE(X,Y,N,1)                                               TAR0733(
      CALL ENDGR (0)                                                    TAR0734(
      RETURN                                                            TAR0735(
      END                                                               TAR0736(
C                                                                       TAR0737(
C                                                                       TAR0738(
      SUBROUTINE PLOT3(YMIN,YMAX,X,Y,N,J,TOTLNG,GSPSG)                  TAR0739(
      IMPLICIT REAL*8(A-H,O-Z)                                          TAR0740(
      REAL*4    X(1),Y(1)                                               TAR0741(
      DO 10 I=1,N                                                       TAR0742(
          X(I) = X(2*I-1)                                               TAR0743(
          Y(I) = Y(2*I-1)                                               TAR0744(
   10 CONTINUE                                                          TAR0745(
      CALL AREA2D(4.,.9)                                                TAR0746(
      CALL XNAME(' ',1)                                                 TAR0747(
      IF(J.EQ.1)CALL YNAME('U$',100)                                    TAR0748(
      IF(J.EQ.2)CALL YNAME('V$',100)                                    TAR0749(
      IF(J.EQ.3)CALL YNAME('W$',100)                                    TAR0750(
      IF(J.EQ.4)CALL YNAME('BETA$',100)                                 TAR0751(
      IF(J.EQ.5)CALL YNAME('OMEGA$',100)                                TAR0752(
      IF(J.EQ.6)CALL YNAME('THETA$',100)                                TAR0753(
      IF(J.EQ.7)CALL YNAME('LAMDA$',100)                                TAR0754(
      CALL XNONUM                                                       TAR0755(
      CALL YAXANG(0.)                                                   TAR0756(
      CALL XTICKS(5)                                                    TAR0757(
      CALL CROSS                                                        TAR0758(
      IF(YMIN.EQ.0. .AND. YMAX.EQ.0.)THEN                               TAR0759(
      YMIN=-1.                                                          TAR0760(
      YMAX=+1.                                                          TAR0761(
      ENDIF                                                             TAR0762(
      IF(YMIN.LT.0. .AND. YMAX.LT.0.)YMAX=0.                            TAR0763(
      IF(YMIN.GT.0. .AND. YMAX.GT.0.)YMIN=0.                            TAR0764(
      CALL GRAF(0.,GSPSG,TOTLNG,YMIN,'SCALE',YMAX)                      TAR0765(
      CALL SCLPIC (.25)                                                 TAR0766(
      CALL MARKER(15)                                                   TAR0767(
      CALL PSPLIN                                                       TAR0768(
      CALL CURVE(X,Y,N,1)                                               TAR0769(
      CALL ENDGR (0)                                                    TAR0770(
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      RETURN                                                                TAR0771C
      END                                                                   TAR0772C
C                                                                           TAR0773C
C                                                                           TAR0774C
      SUBROUTINE STABC(PSM,B,RM,XD,ACC,KDSMAX,IHBBMX,KDS,IHBWB,LIMIT)       TAR0775C
      REAL*8 PSM(KDSMAX,IHBBMX),D1,D2,B(KDSMAX),RM(KDSMAX)                  TAR0776C
     C,A1,A2,SBS,DIFF,ZXM,XD,XM,ACC                                         TAR0777C
      XM=1.                                                                 TAR0778C
      NC=IHBWB-1                                                            TAR0779C
      CALL LUDAPB(PSM,KDS,NC,KDSMAX,PSM,KDSMAX,D1,D2,IER)                   TAR0780C
      WRITE(6,1) D1,D2,IER                                                  TAR0781C
      IF(IER.NE.0)XD=-1.                                                    TAR0782C
      IF(IER.NE.0)RETURN                                                    TAR0783C
      IF(D1.LE.0.0)XD=-1.                                                   TAR0784C
      IF(D1.LE.0.0)RETURN                                                   TAR0785C
      SBS=1.                                                                TAR0786C
      KOUNTY=0                                                              TAR0787C
      DO 40 I=1,KDS                                                         TAR0788C
40    B(I)=1.                                                               TAR0789C
60    CONTINUE                                                              TAR0790C
      ZXM=XM                                                                TAR0791C
      DO 50 I=1,KDS                                                         TAR0792C
      B(I)=B(I)/SBS                                                         TAR0793C
50    RM(I)=B(I)                                                            TAR0794C
      CALL LUELPB(PSM,B,KDS,NC,KDSMAX,B)                                    TAR0795C
      KOUNTY=KOUNTY+1                                                       TAR0796C
      IF(KOUNTY.GT.LIMIT)WRITE(6,919) KOUNTY                                TAR0797C
      IF(KOUNTY.GT.LIMIT)GO TO 9999                                         TAR0798C
      A1=0.                                                                 TAR0799C
      A2=0.                                                                 TAR0800C
      DO 61 I=1,KDS                                                         TAR0801C
      A1=A1+(RM(I)*B(I))                                                    TAR0802C
      A2=A2+(B(I)*B(I))                                                     TAR0803C
61    CONTINUE                                                              TAR0804C
      XM=A1                                                                 TAR0805C
      SBS=DSQRT(A2)                                                         TAR0806C
      IF(KOUNTY.EQ.1)GO TO 60                                               TAR0807C
      DIFF=DABS(ZXM-XM)                                                     TAR0808C
      IF(DIFF.LE.ACC)GO TO 9999                                             TAR0809C
      GO TO 60                                                              TAR0810C
9999  CONTINUE                                                              TAR0811C
      WRITE(6,919) KOUNTY                                                   TAR0812C
919   FORMAT(5X,'NO OF ITERATIONS',I4,/)                                    TAR0813C
      XD=1./XM                                                              TAR0814C
      WRITE(6,888) XD                                                       TAR0815C
888   FORMAT(5X,'SMALLEST EIGEN VALUE',E20.10,/)                            TAR0816C
      RETURN                                                                TAR0817C
1     FORMAT(5X,'DETERMINANT SIGN',2E20.10,I5,//)                           TAR0818C
      END                                                                   TAR0819C
C                                                                           TAR0820C
C                                                                           TAR0821C
      SUBROUTINE PRDCTD(A,B,M,N,L,C)                                        TAR0822C
      REAL*8 A(M,N),B(N,L),C(M,L)                                           TAR0823C
      DO 10 I=1,L                                                           TAR0824C
      DO 20 J=1,M                                                           TAR0825C
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      C(J,I)=0.                                                          TAR0826C
      DO 30 K=1,N                                                        TAR0827C
      C(J,I)=C(J,I)+(A(J,K)*B(K,I))                                      TAR0828C
30    CONTINUE                                                           TAR0829C
20    CONTINUE                                                           TAR0830C
10    CONTINUE                                                           TAR0831C
      RETURN                                                             TAR0832C
      END                                                                TAR0833C
      SUBROUTINE GEOM(SL,G,PJ,OMG,E,EY,B1,ZI,OI,AAA,A,ISS,GAMA)          TAR0834C
      IMPLICIT REAL*8(A-H,O-Z)                                           TAR0835C
      DIMENSION A(ISS,ISS,5)                                             TAR0836C
      DO 10 I=1,ISS                                                      TAR0837C
      DO 10 J=1,ISS                                                      TAR0838C
      DO 10 KK=1,5                                                       TAR0839C
      A(I,J,KK)=0.0                                                      TAR0840C
10    CONTINUE                                                           TAR0841C
      F=DSQRT((G*PJ)/(E*OMG))                                            TAR0842C
      C=DCOSH(F*SL)                                                      TAR0843C
      S=DSINH(F*SL)                                                      TAR0844C
      S2=DSINH(2.*F*SL)                                                  TAR0845C
      C2=DCOSH(2.*F*SL)                                                  TAR0846C
      CONST=-GAMA/ZI                                                     TAR0847C
      A(2,2,1)=(-9./5.)*(SL**5)                                          TAR0848C
      A(2,4,4)=(3.*SL*SL*C)-((6.*SL*S)/F)+((6.*C)/(F*F))-(6./(F*F))      TAR0849C
      A(2,4,5)=(3.*(SL**3)*C)-((9.*SL*SL*S)/F)                           TAR0850C
     &+((18.*SL*C)/(F*F))-((18.*S)/(F*F*F))                              TAR0851C
      A(2,5,4)=(3.*SL*SL*S)-((6.*SL*C)/F)+((6.*S)/(F*F))                 TAR0852C
      A(2,5,5)=(3.*(SL**3)*S)-((9.*SL*SL*C)/F)                           TAR0853C
     &+((18.*SL*S)/(F*F))-((18.*C)/(F*F*F))+(18./(F*F*F))                TAR0854C
      A(2,7,1)=(-1.5)*(SL**4)                                            TAR0855C
      A(2,9,1)=(-SL*SL*SL)                                               TAR0856C
      A(2,11,4)=F*SL*SL*SL                                               TAR0857C
      A(2,11,5)=(3.*F*(SL**4))/4.                                        TAR0858C
      A(3,3,1)=(-9.*(SL**5))/5.                                          TAR0859C
      A(3,4,2)=(3.*SL*SL*C)-((6.*SL*S)/F)+((6.*(C-1))/(F*F))             TAR0860C
      A(3,4,3)=(-3.*(SL**3)*C)+((9.*SL*SL*S)/F)                          TAR0861C
     &+((-18.*SL*C)/(F*F))+((18.*S)/(F*F*F))                             TAR0862C
      A(3,4,1)=(-3.*EY*SL*SL*C)+((6.*SL*EY*S)/F)-((6.*EY*(C-1))/(F*F))   TAR0863C
      A(3,5,2)=(3.*SL*SL*S)-((6.*SL*C)/F)+((6.*S)/(F*F))                 TAR0864C
      A(3,5,3)=(-3.*(SL**3)*S)+((9.*SL*SL*C)/F)-((18.*SL*S)/(F*F))       TAR0865C
     &+((18.*(C-1))/(F*F*F))                                             TAR0866C
      A(3,5,1)=(-3.*EY*SL*SL*S)+((6.*EY*SL*C)/F)-((6.*EY*S)/(F*F))       TAR0867C
      A(3,6,1)=(-1.5)*(SL**4)                                            TAR0868C
      A(3,10,1)=-SL*SL*SL                                                TAR0869C
      A(3,11,2)=(F*SL*SL*SL)                                             TAR0870C
      A(3,11,3)=(-0.75)*F*(SL**4)                                        TAR0871C
      A(3,11,1)=-F*EY*(SL**3)                                            TAR0872C
      A(4,4,1)=((-F*F*OI)/AAA)*((S2/(F*4.))-(SL/2.))                     TAR0873C
      A(4,4,2)=((-F*F*B1)/ZI)*((S2/(F*4.))-(SL/2.))                      TAR0874C
      A(4,4,3)=((F*F*B1)/ZI)*(((SL*S2)/(4.*F))-((SL*SL)/2.)              TAR0875C
     &-((C2-1.)/(8.*F*F))+((SL*SL)/4.))                                  TAR0876C
      A(4,5,1)=((-OI*F*F)/AAA)*((C2-1.)/(4.*F))                          TAR0877C
      A(4,5,2)=((-B1*F*F)/ZI)*((C2-1.)/(4.*F))                           TAR0878C
      A(4,5,3)=((B1*F*F)/ZI)*(((SL*C2)/(F*4.))-(S2/(8.*F*F)))            TAR0879C
      A(4,6,2)=(2.*F)*(((SL*C)/F)-(S/(F*F)))                             TAR0880C
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      A(4,6,3)=(-2.*F)*(((SL*SL*C)/F)+((-2.*SL*S)/(F*F))                TAR0881C
     &+(((C-1.)*2.)/(F*F*F)))                                           TAR0882C
      A(4,6,1)=(-2.*F*EY)*(((SL*C)/F)-(S/(F*F)))                        TAR0883C
      A(4,7,4)=(2.*SL*C)-((2.*S)/F)                                     TAR0884C
      A(4,7,5)=(2.*SL*SL*C)-((4.*SL*S)/F)+((4.*(C-1.))/(F*F))           TAR0885C
      A(4,9,4)=C-1.                                                     TAR0886C
      A(4,9,5)=(SL*C)-(S/F)                                             TAR0887C
      A(4,10,2)=C-1.                                                    TAR0888C
      A(4,10,3)=(-SL*C)+(S/F)                                           TAR0889C
      A(4,10,1)=-EY*(C-1.)                                              TAR0890C
      A(4,11,1)=((-OI*F)/AAA)*(C-1.)                                    TAR0891C
      A(4,11,2)=((-B1*F)/ZI)*(C-1.)                                     TAR0892C
      A(4,11,3)=((B1*F)/ZI)*((SL*C)-(S/F))                              TAR0893C
      A(5,5,1)=((-F*F*OI)/AAA)*((S2/(4.*F))+(SL/2.))                    TAR0894C
      A(5,5,2)=((-F*F*B1)/ZI)*((S2/(4.*F))+(SL/2.))                     TAR0895C
      A(5,5,3)=((F*F*B1)/ZI)*(((SL*S2)/(4.*F))+((SL*SL)/2.)             TAR0896C
     &-((C2-1.)/(8.*F*F))-((SL*SL)/4.))                                 TAR0897C
      A(5,6,2)=(2.*SL*S)-((2.*(C-1.))/F)                                TAR0898C
      A(5,6,3)=(-2.*SL*SL*S)+((4.*SL*C)/F)-((4.*S)/(F*F))               TAR0899C
      A(5,6,1)=(-2.*EY*SL*S)+((2.*EY*(C-1.))/F)                         TAR0900C
      A(5,7,4)=(2.*SL*S)-(2.*(C-1.)/F)                                  TAR0901C
      A(5,7,5)=(2.*SL*SL*S)-((4.*SL*C)/F)+((4.*S)/(F*F))                TAR0902C
      A(5,9,4)=S                                                        TAR0903C
      A(5,9,5)=(SL*S)-((C-1.)/F)                                        TAR0904C
      A(5,10,2)=S                                                       TAR0905C
      A(5,10,3)=(-SL*S)+((C-1.)/F)                                      TAR0906C
      A(5,10,1)=-EY*S                                                   TAR0907C
      A(5,11,1)=(-OI*F*S)/AAA                                           TAR0908C
      A(5,11,2)=(-B1*F*S)/ZI                                            TAR0909C
      A(5,11,3)=((B1*F)/ZI)*((SL*S)-((C-1.)/F))                         TAR0910C
      A(6,6,1)=(-4.*SL*SL*SL)/3.                                        TAR0911C
      A(6,10,1)=-SL*SL                                                  TAR0912C
      A(6,11,2)=F*SL*SL                                                 TAR0913C
      A(6,11,3)=(-2.*F*SL*SL*SL)/3.                                     TAR0914C
      A(6,11,1)=-EY*F*SL*SL                                             TAR0915C
      A(7,7,1)=(-4.*SL*SL*SL)/3.                                        TAR0916C
      A(7,9,1)=-SL*SL                                                   TAR0917C
      A(7,11,4)=F*SL*SL                                                 TAR0918C
      A(7,11,5)=(2.*F*SL*SL*SL)/3.                                      TAR0919C
      A(9,9,1)=-SL                                                      TAR0920C
      A(9,11,4)=F*SL                                                    TAR0921C
      A(9,11,5)=(F*SL*SL)/2.                                            TAR0922C
      A(10,10,1)=-SL                                                    TAR0923C
      A(10,11,1)=-F*EY*SL                                               TAR0924C
      A(10,11,2)=F*SL                                                   TAR0925C
      A(10,11,3)=(-F*SL*SL)/2.                                          TAR0926C
      A(11,11,1)=(-F*F*OI*SL)/AAA                                       TAR0927C
      A(11,11,2)=(-B1*F*F*SL)/ZI                                        TAR0928C
      A(11,11,3)=(B1*F*F*SL*SL)/(2.*ZI)                                 TAR0929C
      A(2,4,5)=A(2,4,5)+(A(2,5,4)/F)                                    TAR0930C
      A(2,5,5)=A(2,5,5)+(A(2,4,4)/F)                                    TAR0931C
      A(2,11,5)=A(2,11,5)+((3.*F*SL*SL*SL*SL)/4.)                       TAR0932C
      A(2,12,5)=A(2,12,5)+(SL*SL*SL)                                    TAR0933C
      A(3,4,3)=A(3,4,3)+(-A(2,5,4)/F)                                   TAR0934C
      A(3,5,3)=A(3,5,3)+(-A(2,4,4)/F)                                   TAR0935C
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      A(3,11,3)=A(3,11,3)+((-3.*F*SL*SL*SL*SL)/4.)                      TAR0936
      A(3,12,3)=A(3,12,3)+(-SL*SL*SL)                                   TAR0937
      A(4,6,3)=A(4,6,3)+(-A(5,6,2)/F)                                   TAR0938
      A(4,7,5)=A(4,7,5)+(A(5,6,2)/F)                                    TAR0939
      A(4,9,5)=A(4,9,5)+(S/F)                                           TAR0940
      A(4,10,3)=A(4,10,3)+(-S/F)                                        TAR0941
      A(5,6,3)=A(5,6,3)+(-A(4,6,2)/F)                                   TAR0942
      A(5,7,5)=A(5,7,5)+(A(4,6,2)/F)                                    TAR0943
      A(5,9,5)=A(5,9,5)+((C-1.)/F)                                      TAR0944
      A(5,10,3)=A(5,10,3)+((1.-C)/F)                                    TAR0945
      A(6,11,3)=A(6,11,3)+((-2.*F*SL*SL*SL)/3.)                         TAR0946
      A(6,12,3)=A(6,12,3)+(-SL*SL)                                      TAR0947
      A(7,11,5)=A(7,11,5)+((2.*F*SL*SL*SL)/3.)                          TAR0948
      A(7,12,5)=A(7,12,5)+(SL*SL)                                       TAR0949
      A(9,11,5)=A(9,11,5)+((F*SL*SL)/2.)                                TAR0950
      A(9,12,5)=A(9,12,5)+SL                                            TAR0951
      A(10,11,3)=A(10,11,3)+((-F*SL*SL)/2.)                             TAR0952
      A(10,12,3)=A(10,12,3)-SL                                          TAR0953
      A(4,4,3)=A(4,4,3)+(A(4,5,1)*((2.*CONST*F)/((-OI*F*F)/AAA)))       TAR0954
      A(4,5,3)=A(4,5,3)+(A(4,4,1)*((CONST*F)/((-OI*F*F)/AAA)))          TAR0955
     C+(A(5,5,1)*((CONST*F)/((-OI*F*F)/AAA)))                           TAR0956
      A(5,5,3)=A(5,5,3)+(A(4,5,1)*((2.*CONST*F)/((-OI*F*F)/AAA)))       TAR0957
      A(4,11,3)=A(4,11,3)+(CONST*S)+(A(4,7,4)*((CONST*F)/2.))           TAR0958
      A(4,12,3)=A(4,12,3)+(CONST*(C-1.))                                TAR0959
      A(5,11,3)=A(5,11,3)+(CONST*(C-1.))+(A(5,6,2)*((CONST*F)/2.))      TAR0960
      A(5,12,3)=A(5,12,3)+(CONST*S)                                     TAR0961
      A(11,11,3)=A(11,11,3)+(CONST*F*F*SL*SL)                           TAR0962
      A(11,12,3)=A(11,12,3)+(CONST*F*SL)                                TAR0963
      RETURN                                                            TAR0964
      END                                                               TAR0965
C                                                                       TAR0966
C                                                                       TAR0967
      SUBROUTINE TRANS(SL,Q,E,PJ,OMG,CC,ISS)                            TAR0968
      IMPLICIT REAL*8(A-H,O-Z)                                          TAR0969
      DIMENSION CC(ISS,ISS)                                             TAR0970
      F=DSQRT((Q*PJ)/(E*OMG))                                           TAR0971
      C=DCOSH(F*SL)                                                     TAR0972
      S=DSINH(F*SL)                                                     TAR0973
      D=(2.*F)-(2.*F*C)+(F*F*SL*S)                                      TAR0974
      DO 10 I=1,ISS                                                     TAR0975
      DO 10 J=1,ISS                                                     TAR0976
      CC(I,J)=0.0                                                       TAR0977
10    CONTINUE                                                          TAR0978
      CC(1,1)=-1./SL                                                    TAR0979
      CC(1,8)=1./SL                                                     TAR0980
      CC(2,2)=2./(SL**3)                                                TAR0981
      CC(2,7)=1./(SL**2)                                                TAR0982
      CC(2,9)=-2./(SL**3)                                               TAR0983
      CC(2,14)=1./(SL**2)                                               TAR0984
      CC(3,3)=2./(SL**3)                                                TAR0985
      CC(3,6)=-1./(SL**2)                                               TAR0986
      CC(3,10)=-2./(SL**3)                                              TAR0987
      CC(3,13)=-1./(SL**2)                                              TAR0988
      CC(4,4)=(F-(F*C))/D                                               TAR0989
      CC(4,5)=((-F*SL*C)+S)/D                                           TAR0990
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```
      CC(4,11)=((-F)+(F*C))/D                                          TAR0991(
      CC(4,12)=(-S+(F*SL))/D                                           TAR0992(
      CC(5,4)=(F*S)/D                                                  TAR0993(
      CC(5,5)=(1.-C+(F*SL*S))/D                                        TAR0994(
      CC(5,11)=(-F*S)/D                                                TAR0995(
      CC(5,12)=(C-1.)/D                                                TAR0996(
      CC(6,3)=-3./(SL**2)                                              TAR0997(
      CC(6,6)=2./SL                                                    TAR0998(
      CC(6,10)=3./(SL**2)                                              TAR0999(
      CC(6,13)=1./SL                                                   TAR1000(
      CC(7,2)=-3./(SL**2)                                              TAR1001(
      CC(7,7)=-2./SL                                                   TAR1002(
      CC(7,9)=3./(SL*SL)                                               TAR1003(
      CC(7,14)=-1./SL                                                  TAR1004(
      CC(8,1)=1.                                                       TAR1005(
      CC(9,7)=1.                                                       TAR1006(
      CC(10,6)=-1.                                                     TAR1007(
      CC(11,4)=(-F*S)/D                                                TAR1008(
      CC(11,5)=(1.-C)/D                                                TAR1009(
      CC(11,11)=(F*S)/D                                                TAR1010(
      CC(11,12)=(1.-C)/D                                               TAR1011(
      CC(12,4)=((F)-(F*C)+(F*F*SL*S))/D                                TAR1012(
      CC(12,5)=((F*SL*C)-S)/D                                          TAR1013(
      CC(12,11)=(F-(F*C))/D                                            TAR1014(
      CC(12,12)=(S-(F*SL))/D                                           TAR1015(
      CC(13,3)=1.                                                      TAR1016(
      CC(14,2)=1.                                                      TAR1017(
      RETURN                                                           TAR1018(
      END                                                              TAR1019(
C                                                                      TAR1020(
C                                                                      TAR1021(
      SUBROUTINE GEMF(I,F,NEMAX,ISS,A,AF)                              TAR1022(
      IMPLICIT REAL*8(A-H,O-Z)                                         TAR1023(
      DIMENSION A(ISS,ISS,5),AF(ISS,ISS),B(5),F(NEMAX,ISS)             TAR1024(
      B(1)=F(I,1)                                                      TAR1025(
      B(2)=F(I,7)                                                      TAR1026(
      B(3)=F(I,2)                                                      TAR1027(
      B(4)=F(I,6)                                                      TAR1028(
      B(5)=F(I,3)                                                      TAR1029(
      DO 10 J=1,ISS                                                    TAR1030(
      DO 20 K=1,ISS                                                    TAR1031(
      STORE=0.0                                                        TAR1032(
      DO 30 L=1,5                                                      TAR1033(
      STORE=STORE+(A(J,K,L)*B(L))                                      TAR1034(
30    CONTINUE                                                         TAR1035(
      AF(J,K)=STORE                                                    TAR1036(
20    CONTINUE                                                         TAR1037(
10    CONTINUE                                                         TAR1038(
      DO 40 KK=1,ISS                                                   TAR1039(
      DO 40 JJ=1,ISS                                                   TAR1040(
      AF(JJ,KK)=AF(KK,JJ)                                              TAR1041(
40    CONTINUE                                                         TAR1042(
      RETURN                                                           TAR1043(
      END                                                              TAR1044(
C                                                                      TAR1045(
```

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C
      SUBROUTINE GLGEMF(C, A, ISS, M)                                   TAR1046(
      IMPLICIT REAL*8(A-H, O-Z)                                         TAR1047(
      DIMENSION A(ISS, ISS), C(ISS, ISS), D(14, 14), E(14, 14)          TAR1048(
      DO 10 I=1, ISS                                                    TAR1049(
      DO 20 J=1, ISS                                                    TAR1050(
      E(I, J)=C(J, I)                                                   TAR1051(
20    CONTINUE                                                          TAR1052(
10    CONTINUE                                                          TAR1053(
      CALL PRDCTD(E, A, ISS, ISS, ISS, D)                               TAR1054(
      CALL PRDCTD(D, C, ISS, ISS, ISS, A)                               TAR1055(
      RETURN                                                            TAR1056(
      END                                                               TAR1057(
                                                                        TAR1058(
```

Appendix F

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